Classical Friedrichs operators in one dimensional scalar case

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PMF-MO

7th Croatian mathematical congress

June $15-18$, 2022, Split, Croatia

Joint work with Marko Erceg

IP-2018-01-2449 (MiTPDE)

Croatian Science Foundation

Assumptions:
$d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary;
$\mathbf{A}_k \in W^{1,\infty}(\Omega; \mathrm{M}_r(\mathbb{C}))$, $k \in \{1, \dots, d\}$, and $\mathbf{B} \in L^\infty(\Omega; \mathrm{M}_r(\mathbb{C}))$ satisfying (a.e. on Ω):

(F1) $$\mathbf{A}_k = \mathbf{A}_k^* \, ;$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \, .$$

Define $\mathcal{L}, \widetilde{\mathcal{L}} : L^2(\Omega)^r \to \mathcal{D}'(\Omega)^r$ by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u} \, , \qquad \widetilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\Big)\mathsf{u} \, .$$

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Aim: impose boundary conditions such that for any $\mathsf{f} \in L^2(\Omega)^r$ we have a unique solution of $\mathcal{L}\mathsf{u} = \mathsf{f}$.
Gain: many important (semi)linear equations of mathematical physics can be written in the form of classical Friedrichs operators.

The classical theory in short

K. O. Friedrichs: *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. **11** (1958) 333–418.

Unified treatment of linear hyperbolic systems like Maxwell's, Dirac's, or higher order equations (e.g. the wave equation).

– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$

– unified treatment of equations and systems of different type;
– more recently: better numerical properties.

Shortcommings:
– no satisfactory well-posedness result,
– no intrinsic (unique) way to pose boundary conditions.

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$\rightsquigarrow$ development of the abstract theory

$(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space ($\mathcal{H}' \equiv \mathcal{H}$), $\| \cdot \| := \sqrt{\langle \cdot \mid \cdot \rangle}$
$\mathcal{D} \subseteq \mathcal{H}$ dense subspace

Definition

Let $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

(T1) $\qquad (\forall \varphi, \psi \in \mathcal{D}) \qquad \langle T\varphi \mid \psi \rangle = \langle \varphi \mid \widetilde{T}\psi \rangle ;$

(T2) $\qquad (\exists c > 0)(\forall \varphi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\varphi\| \leqslant c\|\varphi\| ;$

(T3) $\qquad (\exists \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \qquad \langle (T + \widetilde{T})\varphi \mid \varphi \rangle \geqslant \mu_0 \|\varphi\|^2 .$

A. Ern, J.-L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems*, Comm. Partial Diff. Eq. **32** (2007) 317–341.

N. Antonić, K. Burazin: *Intrinsic boundary conditions for Friedrichs systems*, Comm. Partial Diff. Eq. **35** (2010) 1690–1715.

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$\mathcal{D} := C_c^\infty(\Omega)^r$, $\mathcal{H} := L^2(\Omega)^r$, and

$$T\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u}\,, \qquad \widetilde{T}\mathsf{u} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{B}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k\Big)\mathsf{u}\,.$$

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(T1) $\langle\, T\mathsf{u} \mid \mathsf{v} \,\rangle_{L^2} = \langle\, \mathsf{u} \mid -\sum_{k=1}^d \partial_k(\mathbf{A}_k^* \mathsf{v}) + \big(\mathbf{B}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k\big)\mathsf{v} \,\rangle_{L^2} \overset{(\mathrm{F1})}{=} \langle\, \mathsf{u} \mid \widetilde{T}\mathsf{v} \,\rangle_{L^2}\ .$

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Since $(T + \widetilde{T})\mathsf{u} = \Big(\mathbf{B} + \mathbf{B}^* + \sum_{k=1}^d \partial_k \mathbf{A}_k\Big)\mathsf{u}$,

(T2) $\|(T + \widetilde{T})\mathsf{u}\|_{L^2} \leqslant \Big(2\|\mathbf{B}\|_{L^\infty} + \sum_{k=1}^d \|\mathbf{A}_k\|_{W^{1,\infty}}\Big)\|\mathsf{u}\|_{L^2}\,,$

(T3) $\langle\, (T + \widetilde{T})\mathsf{u} \mid \mathsf{u}\,\rangle_{L^2} \overset{(\mathrm{F2})}{\geqslant} \mu_0 \|\mathsf{u}\|_{L^2}^2\,.$

Characterisation of joint pair of abstract Friedrichs operators

Lemma

$$(T1)-(T3) \iff \begin{cases} T \subseteq \widetilde{T}^* \quad \& \quad \widetilde{T} \subseteq T^*; \\ \overline{T+\widetilde{T}} \text{ bounded self-adjoint in } \mathcal{H} \text{ with strictly positive bottom}; \\ \operatorname{dom}\overline{T} = \operatorname{dom}\overline{\widetilde{T}} \quad \& \quad \operatorname{dom}T^* = \operatorname{dom}\widetilde{T}^*\,. \end{cases}$$

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Dual pairs : Operators A, B on $\mathcal{H}$ with the property that $A \subseteq B^*$ and $B \subseteq A^*$ are often referred to as *dual pairs*. Thus, T and $\widetilde{T}$ are *dual pairs*.

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By (T1) , T and $\widetilde{T}$ are closable. By (T2) , $T+\widetilde{T}$ is a bounded operator, so the graph norms $\|\cdot\|_T$ and $\|\cdot\|_{\widetilde{T}}$ are equivalent.

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$$\begin{aligned} \operatorname{dom} \overline{T} \; &= \; \operatorname{dom} \overline{\widetilde{T}} \; =: \; \mathcal{W}_0 \; , \\ \operatorname{dom} T^* \; &= \; \operatorname{dom} \widetilde{T}^* \; =: \; \mathcal{W} \; , \end{aligned}$$

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and $(\overline{T+\widetilde{T}})|_{\mathcal{W}} = \widetilde{T}^* + T^*$. So, $(\overline{T}, \overline{\widetilde{T}})$ is also a pair of abstract Friedrichs operators.

Notation :

$$T_0 := \overline{T}\,, \quad \widetilde{T}_0 := \overline{\widetilde{T}}\,, \quad T_1 := \widetilde{T}^*\,, \quad \widetilde{T}_1 := T^*\,.$$

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For, $\mathcal{D} = C_c^\infty(\Omega)$, $\mathcal{H} = L^2(\Omega)$ and a certain choice of operators it could be that $\mathcal{W}$ and $\mathcal{W}_0$ are Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively.

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Boundary map (form): $D : \mathcal{W} \to \mathcal{W}'$,

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Let a pair of operators $(T, \widetilde{T})$ on $\mathcal{H}$ satisfies $(\mathrm{T1})$–$(\mathrm{T2})$. Then D is continuous and satisfies

i) $(\forall u, v \in \mathcal{W}) \qquad ([u \,|\, v] = \overline{[v \,|\, u]})$,

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Remark : $(\mathcal{W}, [\cdot \,|\, \cdot])$ is indefinite inner product space.

Well-posedness result

For $\mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W}$ we introduce two conditions:

(V1)
$$(\forall\, u \in \mathcal{V}) \qquad [u \,|\, u] \geqslant 0$$
$$(\forall\, v \in \widetilde{\mathcal{V}}) \qquad [v \,|\, v] \leqslant 0$$

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$$\mathcal{V}^{[\perp]} = \widetilde{\mathcal{V}}\,,\ \widetilde{\mathcal{V}}^{[\perp]} = \mathcal{V}$$

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If $(T, \widetilde{T})$ satisfies (T1)–(T2), then

$$(V2) \quad \Longleftrightarrow \quad \begin{cases} \mathcal{D} \subseteq \mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W} \\ (\widetilde{T}^*|_{\mathcal{V}})^* \;=\; T^*|_{\widetilde{\mathcal{V}}} \\ (T^*|_{\widetilde{\mathcal{V}}})^* \;=\; \widetilde{T}^*|_{\mathcal{V}}\,. \end{cases}$$

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Theorem (Ern, Guermond, Caplain, 2007)

(T1)–(T3) + (V1)–(V2) $\implies T_1|_{\mathcal{V}}, \widetilde{T}_1|_{\widetilde{\mathcal{V}}}$ *bijective realisations .*

We seek for bijective closed operators $S \equiv \widetilde{T}^*|_{\mathcal{V}}$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^* ,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$. We call (S, S^*) an adjoint pair of bijective realisations relative to $(T, \widetilde{T})$.

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Theorem (Antonić, Erceg, Michelangeli, 2017)

Let $(T, \widetilde{T})$ satisfies (T1)–(T3).

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.*

(ii)

$\ker \widetilde{T}^* \neq \{0\}$ & $\ker T^* \neq \{0\} \implies$ *uncountably many adjoint pairs of bijective realisations with signed boundary map*

$\ker \widetilde{T}^* = \{0\}$ *or* $\ker T^* = \{0\} \implies$ *only one adjoint pair of bijective realisations with signed boundary map*

For $(T, \widetilde{T})$ satisfying (T1)–(T3) we have

$$\overline{T} \subseteq \widetilde{T}^* \quad \text{and} \quad \overline{\widetilde{T}} \subseteq T^* ,$$

while by the previous theorem there exists closed T_{r} such that

- $\overline{T} \subseteq T_{\mathrm{r}} \subseteq \widetilde{T}^*$ ($\iff$ $\overline{\widetilde{T}} \subseteq T_{\mathrm{r}}^* \subseteq T^*$),
- $T_{\mathrm{r}} : \operatorname{dom} T_{\mathrm{r}} \to \mathcal{H}$ bijection,
- $(T_{\mathrm{r}})^{-1} : \mathcal{H} \to \operatorname{dom} T_{\mathrm{r}}$ bounded.

Thus, we can apply a universal classification (classification of dual (adjoint) pairs).

We used Grubb's universal classification

G. Grubb: *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (1968) 425–513.

N. Antonić, M.E., A. Michelangeli: *Friedrichs systems in a Hilbert space framework: solvability and multiplicity*, J. Differential Equations 263 (2017) 8264-8294.

Result: complete classification of all adjoint pairs of bijective realisations with signed boundary map.
To do: apply this result to general classical Friedrichs operators from the beginning (*nice class of non-self-adjoint differential operators of interest*)

(P1) **Grubb's decomposition :**

$$\operatorname{dom} T_1 = \operatorname{dom} T_r \dot{+} \ker T_1 \ ,$$
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(P2) $(\mathcal{W}, [\cdot \mid \cdot])$ is indefinite inner product space.

$$\mathcal{W}_0 \subseteq \mathcal{V} \subseteq \mathcal{W} \text{ is closed in } \mathcal{W} \iff \mathcal{V} = \mathcal{V}^{[\perp][\perp]}.$$

Some preliminary results

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$$(\forall u \in \mathcal{V}) \quad |\langle T_1 u | u \rangle| \geq \mu_0 \|u\|^2 \,,$$
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$$\mathcal{H} = \operatorname{ran} T_0 \oplus \ker \widetilde{T}_1 = \operatorname{ran} \widetilde{T}_0 \oplus \ker T_1 \ .$$

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(P3) If $\mathcal{V}, \widetilde{\mathcal{V}} \subset \mathcal{W}$ and $(\mathcal{V}, \widetilde{\mathcal{V}})$ satisfies the condition (V1) then

$$(\forall u \in \mathcal{V}) \quad |\langle T_1 u | u \rangle| \geq \mu_0 \|u\|^2 \,,$$
$$(\forall v \in \widetilde{\mathcal{V}}) \quad |\langle \widetilde{T}_1 v | v \rangle| \geq \mu_0 \|v\|^2 \ .$$

(P4)

$$\mathcal{H} = \operatorname{ran} T_0 \oplus \ker \widetilde{T}_1 = \operatorname{ran} \widetilde{T}_0 \oplus \ker T_1 \ .$$

(P5) $(\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1, \mathcal{W}_0 \dot{+} \ker T_1)$ satisfies (V1) condition.

Decomposition of the graph space

Theorem

$(T_0, \widetilde{T}_0)$ *is a joint pair of closed abstract Friedrichs operators then*

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$\Omega = (a, b)$, $a < b$, $\mathcal{D} = C_c^\infty(a, b)$ and $\mathcal{H} = L^2(a, b)$. $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$:

$$T\varphi := (\alpha\varphi)' + \beta\varphi \qquad \text{and} \qquad \widetilde{T}\varphi := -(\alpha\varphi)' + (\overline{\beta} + \alpha')\varphi \,.$$

Here $\alpha \in W^{1,\infty}((a, b); \mathbb{R})$, $\beta \in L^\infty((a, b); \mathbb{C})$ and for some $\mu_0 > 0$, $2\Re\beta + \alpha' \geq 2\mu_0 > 0$.

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Lemma

Let $I := [a, b] \setminus \alpha^{-1}(\{0\})$. Then $\mathcal{W} \subseteq H^1_{\mathrm{loc}}(I)$, i.e. for any $u \in \mathcal{W}$ and $[c, d] \subseteq I$, $c < d$, we have $u|_{[c,d]} \in H^1(c, d)$.

The boundary operator can be written explicitly as

$$_{\mathcal{W}'}\langle\, Du, v \,\rangle_{\mathcal{W}} = \big(\alpha u\overline{v}\big)(b) - \big(\alpha u\overline{v}\big)(a)\;, \quad u, v \in \mathcal{W}\;,$$

where we define

$$\big(\alpha u\overline{v}\big)(x) := \begin{cases} 0 & , \quad \alpha(x) = 0 \\ \alpha(x)u(x)\overline{v(x)} & , \quad \alpha(x) \neq 0 \end{cases} \;, \quad x \in [a, b]\,.$$

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$$\dim(\mathcal{W}/\mathcal{W}_0) = \begin{cases} 2 & , \quad \alpha(a)\alpha(b) \neq 0\;, \\ 1 & , \quad \big(\alpha(a) = 0 \wedge \alpha(b) \neq 0\big) \vee \big(\alpha(a) \neq 0 \wedge \alpha(b) = 0\big)\;, \\ 0 & , \quad \alpha(a) = \alpha(b) = 0\;. \end{cases}$$

By the decomposition we have

$$\dim(\ker T_1) + \dim(\ker \widetilde{T}_1) = \dim \mathcal{W}/\mathcal{W}_0 \,.$$

Thus, when $\alpha(a)\alpha(b) = 0$ there is only one bijective realisation of T_0. When case $\alpha(a)\alpha(b) \neq 0$ there are infinitely many bijective realisations if and only if $\dim(\ker T_1) = \dim(\ker \widetilde{T}_1)$.

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The only interesting case is, when $\alpha(a) > 0, \alpha(b) > 0$. In this case we have, $u \in \mathcal{W}$ belongs to $\operatorname{dom} T_{c,d}$ if and only if

[1] $$\left(\frac{\alpha(b)\overline{\tilde{\varphi}(b)}}{\|\tilde{\varphi}\|^2} - \frac{(c+id)}{\varphi(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)} \right) u(b) = \left(\frac{\alpha(a)\overline{\tilde{\varphi}(a)}}{\|\tilde{\varphi}\|^2} - \frac{(c+id)\sqrt{\frac{\alpha(a)}{\alpha(b)}}}{\varphi(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)} \right) u(a) \,.$$

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Similarly, $u \in \mathcal{W}$ is in $\operatorname{dom} T^*_{c,d}$ if and only if

$$[2] \left(\alpha(b)\overline{\varphi(b)} - \frac{\|\tilde{\varphi}\|^2(c-id)}{\tilde{\varphi}(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\tilde{\varphi}(a)} \right) u(b) = \left(\alpha(a)\overline{\varphi(a)} - \frac{\|\tilde{\varphi}\|^2(c-id)\sqrt{\frac{\alpha(a)}{\alpha(b)}}}{\tilde{\varphi}(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\tilde{\varphi}(a)} \right) u(a) \,.$$

So, the set of all pairs of mutually adjoint bijective realisations relative to $(T, \widetilde{T})$ is given by

[3] $$\Big\{(T_{c,d}, T^*_{c,d}) : c, d \in \mathbb{R}^2 \setminus \{(0,0)\}\Big\} \bigcup \big\{(T_{\mathrm{r}}, T^*_{\mathrm{r}})\big\} \; .$$

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Summary :

α at end-points	No. of bij. realisations	$(\mathcal{V}, \widetilde{\mathcal{V}})$	
$\alpha(a)\alpha(b) \leq 0$	1	$\alpha(a) \geq 0 \wedge \alpha(b) \leq 0$	$(\mathcal{W}_0, \mathcal{W})$
		$\alpha(a) \leq 0 \wedge \alpha(b) \geq 0$	$(\mathcal{W}, \mathcal{W}_0)$
$\alpha(a)\alpha(b) > 0$	∞	[3] (see [1] and [2])	

...thank you for your attention :)