Abstract Friedrichs operators and skew self-adjoint realisations

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Joint work with Marko Erceg

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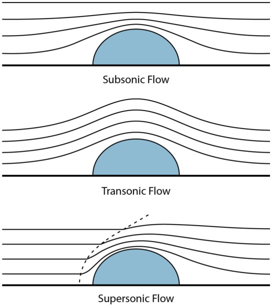
Motivation

- Unified treatment of linear hyperbolic systems like Maxwell's, Dirac's, or higher order equations (e.g. the wave equation).
- Treating the equations of mixed type, such as the Tricomi equation (transonic flow)

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,;$$



Streamlines for three airflow regimes black lines around a nondescript blunt (blue) body.

- unified treatment of equations and systems of different types.

Outline

Classical Friedrichs operators

Assumptions:
$d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary;
$\mathsf{A}_k \in W^{1,\infty}(\Omega; \mathrm{M}_r(\mathbb{C}))$, $k \in \{1, \dots, d\}$, and $\mathsf{B} \in L^\infty(\Omega; \mathrm{M}_r(\mathbb{C}))$ satisfying (a.e. on Ω):

$$\mathsf{A}_k = \mathsf{A}_k^* \,; \tag{F1}$$

$$(\exists\, \mu_0 > 0) \quad \mathsf{B} + \mathsf{B}^* + \sum_{k=1}^{d} \partial_k \mathsf{A}_k \geqslant 2\mu_0 \mathsf{I} \,. \tag{F2}$$

Define $\mathcal{L}, \widetilde{\mathcal{L}} : L^2(\Omega)^r \to \mathcal{D}'(\Omega)^r$ by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathsf{A}_k \mathsf{u}) + \mathsf{B}\mathsf{u} \,, \qquad \widetilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathsf{A}_k \mathsf{u}) + \Big(\mathsf{B}^* + \sum_{k=1}^{d} \partial_k \mathsf{A}_k\Big)\mathsf{u} \,.$$

$\mathcal{L}$ (as well $\widetilde{\mathcal{L}}$) is called *Classical Friedrichs operator* .

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Aim: impose boundary conditions such that for any $\mathsf{f} \in L^2(\Omega)^r$ we have a unique solution of $\mathcal{L}\mathsf{u} = \mathsf{f}$.

K. O. Friedrichs: *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. **11** (1958) 333–418.

Some examples

- Tricomi equation and Generalised Tricomi equations (Frankl equations).
- Scalar elliptic equations.
- First and Second order hyperbolic systems.
- Maxwell's equations in the diffusive regime.
- Stationary diffusion equation.
- Dirac system.
- Dirac-Klein-Gordon systems.
- Maxwell-Dirac system.
- Time-harmonic Maxwell system.

M. Jensen: *Discontinuous Galerkin methods for Friedrichs systems with irregular solutions*, Ph.D. thesis, University of Oxford, 2004, http://sro.sussex.ac.uk/45497/1/thesisjensen.pdf

N. Antonić, K. Burazin, I. Crnjac, M. Erceg: *Complex Friedrichs systems and applications*, *J. Math. Phys.* **58** (2017) 101508.

Abstract Friedrichs operators

$(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space ($\mathcal{H}' \equiv \mathcal{H}$), $\| \cdot \| := \sqrt{\langle \cdot \mid \cdot \rangle}$
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Definition (Ern, Guermond, Caplain, 2007)

Let $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

$$(\forall \varphi, \psi \in \mathcal{D}) \qquad \langle T\varphi \mid \psi \rangle = \langle \varphi \mid \widetilde{T}\psi \rangle \,; \tag{T1}$$

$$(\exists\, c > 0)(\forall \varphi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\varphi\| \leqslant c\|\varphi\| \,; \tag{T2}$$

$$(\exists\, \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \qquad \langle (T + \widetilde{T})\varphi \mid \varphi \rangle \geqslant \mu_0\|\varphi\|^2 \,. \tag{T3}$$

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- Hilbert space theory (beyond PDEs).

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- Avoids invoking traces at the boundary (intrinsic way to impose boundary conditions).

By (T1) , T and $\widetilde{T}$ are closable.
By (T2) , $T+\widetilde{T}$ is a bounded operator, so the graph norms $\|\cdot\|_T$ and $\|\cdot\|_{\widetilde{T}}$ are equivalent.

$$\begin{aligned} \operatorname{dom}\overline{T} &= \operatorname{dom}\overline{\widetilde{T}} =: \mathcal{W}_0 \,, \\ \operatorname{dom}T^* &= \operatorname{dom}\widetilde{T}^* =: \mathcal{W} \,, \end{aligned}$$

and $(\overline{T+\widetilde{T}})|_{\mathcal{W}} = \widetilde{T}^* + T^*$. So, $(\overline{T},\overline{\widetilde{T}})$ is also a pair of abstract Friedrichs operators.

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Notation :

$$T_0 := \overline{T}\,, \quad \widetilde{T}_0 := \overline{\widetilde{T}}\,, \quad T_1 := \widetilde{T}^*\,, \quad \widetilde{T}_1 := T^*\,.$$

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Therefore, we have

$$T_0 \;\subseteq\; T_1 \quad \text{and} \quad \widetilde{T}_0 \;\subseteq\; \widetilde{T}_1 \;.$$

Graph space and boundary map

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Boundary map (form): $D : \mathcal{W} \to \mathcal{W}'$,

$$[u \mid v] := {}_{\mathcal{W}'}\langle Du, v\rangle_{\mathcal{W}} := \langle\, T_1 u \mid v \,\rangle - \langle\, u \mid \widetilde{T}_1 v \,\rangle\,.$$

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Let a pair of operators $(T, \widetilde{T})$ on $\mathcal{H}$ satisfies (T1)–(T2). Then D is continuous and satisfies

i) $(\forall u, v \in \mathcal{W}) \qquad [u \mid v] = \overline{[v \mid u]}\,,$

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Remark: $(\mathcal{W}, [\cdot \,|\, \cdot])$ is indefinite inner product space and $(\mathcal{W} \setminus \ker D, [\cdot \,|\, \cdot])$ is a Kreĭn space.

Well-posedness result

Cone formalism: For $\mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W}$ we introduce two conditions:

(V1)
$$(\forall\, u \in \mathcal{V}) \qquad [u \mid u] \geqslant 0$$
$$(\forall\, v \in \widetilde{\mathcal{V}}) \qquad [v \mid v] \leqslant 0$$

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$$\mathcal{V}^{[\perp]} = \widetilde{\mathcal{V}}\,,\ \widetilde{\mathcal{V}}^{[\perp]} = \mathcal{V}\,.$$

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We seek for bijective closed operators $S \equiv T_1|_{\mathcal{V}}$ such that

$$T_0 \subseteq S \subseteq T_1\,,$$

and thus also S^* is bijective and $\widetilde{T}_0 \subseteq S^* \subseteq \widetilde{T}_1$.
We call (S, S^*) an adjoint pair of bijective realisations relative to $(T, \widetilde{T})$.

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Theorem (Ern, Guermond, Caplain, 2007)

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Theorem (Antonić, Erceg, Michelangeli, 2017)

Let $(T, \widetilde{T})$ satisfies (T1)–(T3).

(i) *Existence: There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.*

(ii) *Multiplicity:*

$\ker \widetilde{T}^* \neq \{0\}$ & $\ker T^* \neq \{0\}$ $\Longrightarrow$ *uncountably many adjoint pairs of bijective realisations with signed boundary map*

$\ker \widetilde{T}^* = \{0\}$ *or* $\ker T^* = \{0\}$ $\Longrightarrow$ *only one adjoint pair of bijective realisations with signed boundary map*

Existence, multiplicity and classification

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Classification: $T_0 \subseteq T_1$, $\widetilde{T}_0 \subseteq \widetilde{T}_1$ and there exists a bijection $T_r : \operatorname{dom} T_{\rm r} \to \mathcal{H}$ with bounded inverse and

$$T_0 \subseteq T_{\rm r} \subseteq T_1 \;(\iff\; \widetilde{T}_0 \subseteq T_{\rm r}^* \subseteq \widetilde{T}_1)\,.$$

Thus, we can apply a universal classification (classification of dual (adjoint) pairs).

N. Antonić, M. Erceg, A. Michelangeli: *Friedrichs systems in a Hilbert space framework: solvability and multiplicity*, *J. Differ. Equ.* **263** (2017) 8264–8294.

G. Grubb: *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (1968) 425–513.

Theorem (Decomposition of the graph space)

$(T_0, \widetilde{T}_0)$ *is a joint pair of closed abstract Friedrichs operators then*

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Decomposition

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Corollary

$\left(T_1|_{\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1}, \widetilde{T}_1|_{\mathcal{W}_0 \dot{+} \ker T_1} \right)$ *is a pair of mutually adjoint pair of bijective realisations relative to* $(T, \tilde{T})$.

M. Erceg, S.K. Soni: *Classification of classical Friedrichs differential operators: One-dimensional scalar case, Commun. Pure Applied Analysis* **10** (2022) 3499–3527. https://doi.org/10.3934/cpaa.2022112

Theorem

Let $(T_0, \widetilde{T}_0)$ be a joint pair of closed abstract Friedrichs operators on $\mathcal{H}$. There exists a subspace $\mathcal{V}$ of $\mathcal{W}$ with $\mathcal{W}_0 \subseteq \mathcal{V}$, such that $(T_1|_{\mathcal{V}}, \widetilde{T}_1|_{\mathcal{V}})$ is a pair of mutually adjoint bijective realisations related to $(T_0, \widetilde{T}_0)$ if and only if $\ker T_1$ and $\ker \widetilde{T}_1$ are isomorphic.

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von-Neumann type decomposition formula: Let $U : (\ker \widetilde{T}_1, [\cdot \mid \cdot]) \to (\ker T_1, -[\cdot \mid \cdot])$ be a unitary transformation, then such $\mathcal{V}$ is given by

$$\mathcal{V} := \left\{ w_0 + U\tilde{\nu} + \tilde{\nu} : w_0 \in \mathcal{W}_0,\ \tilde{\nu} \in \ker \widetilde{T}_1 \right\} . \qquad (1)$$

Conclusion

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Let $(T_0, \widetilde{T}_0)$ *be a joint pair of closed abstract Friedrichs operators on* $\mathcal{H}$*. Let* $(T_1|_{\mathcal{V}}, \widetilde{T}_1|_{\widetilde{\mathcal{V}}})$ *be a pair of mutually adjoint realisations related to* $(T_0, \widetilde{T}_0)$*. Then,* $T_1|_{\mathcal{V}}$ *can be written as the sum of a bounded self-adjoint operator and a skew self-adjoint (possibly unbounded) operator if and only if* $\mathcal{V} = \widetilde{\mathcal{V}}$*.*
Furthermore, if the decomposition above holds, then it is given by

$$T_1|_{\mathcal{V}} = \frac{1}{2}(\overline{T_0 + \widetilde{T}_0}) + \frac{1}{2}(T_1 - \widetilde{T}_1)|_{\mathcal{V}} \ ,$$

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A consequence of the previous theorem is that the search for all pairs $(\mathcal{V}, \mathcal{V})$ is equivalent to the search for all skew self-adjoint realisations of the operator $T_0 - \widetilde{T}_0$, or self-adjoint realisations of the operator $i(T_0 - \widetilde{T}_0)$.

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What's next ...

- Spectral properties, Weyl m-functions and Kreĭn resolvent formula via boundary triplets.
- Semigroup theory.

Reference

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