Classification of boundary conditions for Friedrichs systems

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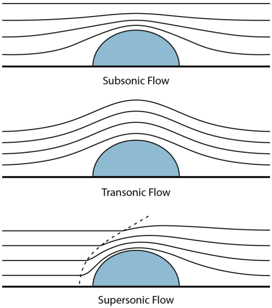

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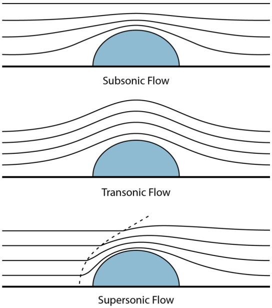

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- unified treatment of equations and systems of different types.

Outline

Classical Friedrichs operators

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Define $\mathcal{L}, \widetilde{\mathcal{L}} : L^2(\Omega)^r \to \mathcal{D}'(\Omega)^r$ by

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Aim: impose boundary conditions such that for any $\mathrm{f} \in L^2(\Omega)^r$ we have a unique solution of $\mathcal{L}\mathrm{u} = \mathrm{f}$.

K. O. Friedrichs: *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. **11** (1958) 333–418.

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The equation is symmetric, but not positive (because of change of sign of y).

Example cont...

We multiply by

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Here,

$$B + B^* + \partial_x A_1 + \partial_y A_2 = \begin{bmatrix} 1+2\lambda y & 2\lambda y \\ 2\lambda y & 2\lambda \end{bmatrix}$$

is positive definite for small λ.

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$$\mathsf{A}_\nu := \sum_{k=1}^{d} \nu_k \mathsf{A}_k \in L^\infty(\Gamma; \mathsf{M}_r),$$

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for given M–admissible boundary condtion—Friedrichs (for a.e $x \in \Gamma$)

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prescribed boundary condition

$$(\mathsf{A}_\nu - \mathsf{M})\mathsf{u}|_\Gamma = 0\ .$$

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- Shortcommings:
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  - no intrinsic (unique) way to pose boundary conditions.

$\rightsquigarrow$ development of the abstract theory

Abstract Friedrichs operators

$(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space $(\mathcal{H}' \equiv \mathcal{H})$, $\| \cdot \| := \sqrt{\langle \cdot \mid \cdot \rangle}$
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Definition (Ern, Guermond, Caplain, 2007)

Let $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

$$(\forall\, \varphi, \psi \in \mathcal{D}) \qquad \langle\, T\varphi \mid \psi \,\rangle = \langle\, \varphi \mid \widetilde{T}\psi \,\rangle\,; \tag{T1}$$

$$(\exists\, c > 0)(\forall\, \varphi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\varphi\| \leqslant c\|\varphi\|\,; \tag{T2}$$

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- A set of geometric conditions (cone-formalism) to ensure well-posedness.

Classical is abstract

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$\mathcal{D} := C_c^\infty(\Omega)^r$, $\mathcal{H} := L^2(\Omega)^r$, and $T\mathsf{u} := \mathcal{L}\mathsf{u}$, $\widetilde{T}\mathsf{u} := \widetilde{\mathcal{L}}\mathsf{u}$.

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Classical is abstract

$d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary;
$\mathrm{A}_k \in W^{1,\infty}(\Omega; \mathrm{M}_r(\mathbb{C}))$, $k \in \{1, \dots, d\}$, and $\mathrm{B} \in L^\infty(\Omega; \mathrm{M}_r(\mathbb{C}))$ satisfying (a.e. on Ω):

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Characterisation of joint pair abstract Friedrichs operators

Lemma

$$(T1)-(T3) \iff \begin{cases} T \subseteq \widetilde{T}^* \quad \& \quad \widetilde{T} \subseteq T^*; \\ \overline{T+\widetilde{T}} \textit{ bounded self-adjoint in } \mathcal{H} \textit{ with strictly positive bottom}; \\ \operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} \quad \& \quad \operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* . \end{cases}$$

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By (T1) , T and $\widetilde{T}$ are closable. By (T2) , $T+\widetilde{T}$ is a bounded operator, so the graph norms $\|\cdot\|_T$ and $\|\cdot\|_{\widetilde{T}}$ are equivalent.

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and $(\overline{T+\widetilde{T}})|_{\mathcal{W}} = \widetilde{T}^* + T^*$. So, $(\overline{T}, \overline{\widetilde{T}})$ is also a pair of abstract Friedrichs operators.

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Remark: $(\mathcal{W}, [\cdot \mid \cdot])$ is indefinite inner product space.

Well-posedness result

For $\mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W}$ we introduce two conditions:

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$$(\forall\, u \in \mathcal{V}) \qquad [u \mid u] \geqslant 0$$
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Scalar elliptic PDE

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$$\begin{aligned} -\triangle u + \mu u = f \iff & -\operatorname{div} \nabla u + \mu u = f \\ \iff & \begin{cases} \nabla u + \mathsf{p} = 0 \\ \operatorname{div} \mathsf{p} + \mu u = f \end{cases} \\ \iff & T\mathsf{v} := \sum_{k=1}^{d} \partial_k(\mathsf{A}_k \mathsf{v}) + \mathsf{B}\mathsf{v} = \mathsf{g}\,, \end{aligned}$$

where $\mathsf{v} := [\mathsf{p}\ u]^\top$, $\mathsf{g} := [0\ f]^\top$, $(\mathsf{A}_k)_{ij} := \delta_{i,k}\delta_{j,d+1} + \delta_{i,d+1}\delta_{j,k}$, $\mathsf{B} := \operatorname{diag}\{1, \ldots, 1, \mu\}$.
Assumtions (F1) and (F2) are satisfied.

$\mathcal{H} = L^2(\Omega)^{d+1}$, $\mathcal{W} = L^2_{\mathrm{div}}(\Omega) \times H^1(\Omega)$ and $\mathcal{W}_0 = L^2_{\mathrm{div},0}(\Omega) \times H^1_0(\Omega)$

- $\mathcal{V} = L^2_{\mathrm{div}}(\Omega) \times H^1_0(\Omega)$ ... Dirichelt boundary condition ($u = 0$ on Γ)
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If $(T, \widetilde{T})$ satisfies (T1)–(T2), then

$$(V2) \quad \Longleftrightarrow \quad \begin{cases} \mathscr{D} \subseteq \mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W} \\ \mathcal{V} \text{ and } \widetilde{\mathcal{V}} \text{ closed in } \mathcal{W} \\ (T_1|_{\mathcal{V}})^* \; = \; \widetilde{T}_1|_{\widetilde{\mathcal{V}}} \\ (\widetilde{T}_1|_{\widetilde{\mathcal{V}}})^* \; = \; T_1|_{\mathcal{V}} \, . \end{cases}$$

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and thus also S^* is bijective and $\widetilde{T}_0 \subseteq S^* \subseteq \widetilde{T}_1$. We call (S, S^*) an adjoint pair of bijective realisations relative to $(T, \widetilde{T})$.

Theorem (Antonić, Erceg, Michelangeli, 2017)

Let $(T, \widetilde{T})$ satisfies (T1)–(T3).

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.*

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$$\ker \widetilde{T}^* \neq \{0\} \ \& \ \ker T^* \neq \{0\} \implies \text{uncountably many adjoint pairs of bijective realisations with signed boundary map}$$

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Thus, we can apply a universal classification (classification of dual (adjoint) pairs).

Abstract theory

N. Antonić, M. Erceg, A. Michelangeli: *Friedrichs systems in a Hilbert space framework: solvability and multiplicity*, *J. Differ. Equ.* **263** (2017) 8264–8294.

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To sum up: in abstract theory, we have

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goal: further study of classical theory, using the results from the abstract theory.

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6. Some applications to PDEs of interest.

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$(T_0, \widetilde{T}_0)$ *is a joint pair of closed abstract Friedrichs operators then*

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Lemma

Let $I := [a, b] \setminus \alpha^{-1}(\{0\})$. Then $\mathcal{W} \subseteq H^1_{\mathrm{loc}}(I)$, i.e. for any $u \in \mathcal{W}$ and $[c, d] \subseteq I$, $c < d$, we have $u|_{([c,d])} \in H^1(c, d)$.

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$$\dim(\mathcal{W}/\mathcal{W}_0) = \begin{cases} 2 & , \quad \alpha(a)\alpha(b) \neq 0\ , \\ 1 & , \quad \big(\alpha(a) = 0 \land \alpha(b) \neq 0\big) \lor \big(\alpha(a) \neq 0 \land \alpha(b) = 0\big)\ , \\ 0 & , \quad \alpha(a) = \alpha(b) = 0\ . \end{cases}$$

1d scalar case cont...

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Summary :

α at end-points	No. of bij. realisations	$(\mathcal{V}, \widetilde{\mathcal{V}})$	
$\alpha(a)\alpha(b) \leq 0$	1	$\alpha(a) \geq 0 \wedge \alpha(b) \leq 0$	$(\mathcal{W}_0, \mathcal{W})$
		$\alpha(a) \leq 0 \wedge \alpha(b) \geq 0$	$(\mathcal{W}, \mathcal{W}_0)$
$\alpha(a)\alpha(b) > 0$	∞	explicit formulae	

1d vectorial case

$\Omega = (a, b),\ a < b$. Then $\mathscr{D} = C_c^\infty((a,b),\mathbb{C}^r)$ and $\mathscr{H} = L^2((a,b),\mathbb{C}^r)$.

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$\Omega = (a, b)$, $a < b$. Then $\mathscr{D} = C_c^\infty((a, b), \mathbb{C}^r)$ and $\mathscr{H} = L^2((a, b), \mathbb{C}^r)$. $T, \widetilde{T} : \mathscr{D} \to \mathscr{H}$:

$$T\mathsf{u} := (\mathsf{A}\mathsf{u})' + \mathsf{B}\mathsf{u} \qquad \text{and} \qquad \widetilde{T}\mathsf{u} := -(\mathsf{A}\varphi)' + (\mathsf{B}^* + \mathsf{A}')\mathsf{u} \ ,$$

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where $\mathsf{A} \in W^{1,\infty}((a,b); \mathrm{M}_r)$, $\mathsf{B} \in L^\infty((a,b); \mathbb{C}^r)$ and for some $\mu_0 > 0$ we have $\mathsf{B}^* + \mathsf{B} + \mathsf{A}' \geq 2\mu_0 \mathsf{I} > 0$.

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The graph space:

$$\mathcal{W} \;=\; \left\{\mathsf{u} \in \mathcal{H} : (\mathsf{A}\mathsf{u})' \in \mathcal{H}\right\} = \left\{\mathsf{u} \in \mathcal{H} : \mathsf{A}\mathsf{u} \in H^1(a,b)\right\} .$$

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A is diagonalizable, $\mathrm{A} = \mathrm{Q}\Lambda\mathrm{Q}^*$, orthogonal matrix $\mathrm{Q} = [\mathrm{v}_1, ..., \mathrm{v}_r]^T$, $\Lambda = \mathrm{diag}[\lambda_1, ..., \lambda_r]$.

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$$\forall \mathrm{u},\mathrm{v} \in \mathscr{W} : [\,\mathrm{u} \mid \mathrm{v}\,] = (\Lambda\hat{\mathrm{u}}\cdot\hat{\mathrm{v}})(b) - (\Lambda\hat{\mathrm{u}}\cdot\hat{\mathrm{v}})(a) = \sum_{k=1}^{r}\big(\lambda_k(b)\hat{u}_k(b)\overline{\hat{v}}_k(b) - \lambda_k(a)\hat{u}_k(a)\overline{\hat{v}}_k(a)\big) \ .$$

And, $\quad \mathscr{W}_0 = \{u \in \mathscr{W} : (\Lambda\hat{\mathrm{u}})(b) = (\Lambda\hat{\mathrm{v}})(a) = 0\}$.

Construction of a pair $(\mathcal{V}, \widetilde{\mathcal{V}})$

We first define the subspaces $\{\mathcal{V}_{k,j}\}_{k=1,..,r}^{j=1,2}$ and $\{\widetilde{\mathcal{V}}_{k,j}\}_{k=1,\dots,r}^{j=1,2}$ of $\mathcal{W}$

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Sign of $\lambda_k(a)$	$\mathcal{V}_{k,1}$	$\widetilde{\mathcal{V}}_{k,1}$
$\lambda_k(a) = 0$	$\mathcal{W}$	$\mathcal{W}$
$\lambda_k(a) > 0$	$\{\mathsf{u} \in \mathcal{W} : \hat{u}_k(a) = 0\}$	$\mathcal{W}$
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Sign of $\lambda_k(b)$	$\mathcal{V}_{k,2}$	$\widetilde{\mathcal{V}}_{k,2}$
$\lambda_k(b) = 0$	$\mathcal{W}$	$\mathcal{W}$
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$\lambda_k(b) = 0$	$\mathcal{W}$	$\mathcal{W}$
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Define,

$$\mathcal{V} := \bigcap_{k=1}^{r} \bigcap_{j=1}^{2} \mathcal{V}_{k,j} \quad \text{and} \quad \widetilde{\mathcal{V}} := \bigcap_{k=1}^{r} \bigcap_{j=1}^{2} \widetilde{\mathcal{V}}_{k,j} \ .$$

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Lemma

$(\mathcal{V}, \widetilde{\mathcal{V}})$ satisfy the conditions $(V1) - (V2)$.

Result on kernels

Theorem

$$\dim \ker T_1 = n_a^+ + n_b^- \text{ and } \dim \ker \widetilde{T}_1 = n_a^- + n_b^+$$

Where, n_a^+, n_a^- are the number of positive and negative eigenvalues of the matrix $\mathrm{A}(a)$ *respectively and similarly, n_b^+, n_b^- are the number of positive and negative eigenvalues of the matrix* $\mathrm{A}(b)$ *respectively.*

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Corollary

$$\dim\left(\mathcal{W}/\mathcal{W}_0\right) = \operatorname{rank}(\mathrm{A}(a)) + \operatorname{rank}(\mathrm{A}(b)) \,.$$

Lemma

For existence of $\mathcal{V} = \widetilde{\mathcal{V}}$, we have

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Here, n_a^0, n_b^0 are the number of zero eigenvalues of $A(a)$ and $A(b)$ respectively.

$\mathcal{V} = \widetilde{\mathcal{V}}$

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$(\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1, \mathcal{W}_0 \dot{+} \ker T_1)$ is an admissible pair.

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$(\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1, \mathcal{W}_0 \dot{+} \ker T_1)$ is an admissible pair. So, for any other pair $(\mathcal{V}, \widetilde{\mathcal{V}})$, we have

$$\mathcal{V}/\mathcal{W}_0 \cong \ker T_1, \text{ and } \widetilde{\mathcal{V}}/\mathcal{W}_0 \cong \ker \widetilde{T}_1 .$$

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For the other part we follow the construction.

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