H-distributions with a characteristic length

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Weak convergences and partial differential equations

Suppose we want to solve (possibly nonlinear) equation: $\mathcal{A}[u] = f$.
Here, $\mathcal{A}$ is some complicated partial differential operator, and the equation contains some additional conditions (boundary and/or initial).

We might try the following procedure:
Approximate $\mathcal{A}$ by a sequence $\mathcal{A}_n$ of operators we know how to solve, and also f by a sequence f_n of nicer functions, if needed.

Then solve each of the problems: $\mathcal{A}_n[u_n] = f_n$, obtaining the solutions u_n.

It is only natural to expect that the limit $u := \lim u_n$ will be a solution of the original problem.

Of course, this is only a rough idea — in each particular case we have to be more precise. In particular with the definition of various limits taken.

Weak convergences and defect measures

In the above procedure, one usually only gets weakly converging sequences

$$u_n \rightharpoonup u$$

in some L^p space.

However, one cannot just pass to the weak limit with a nonlinear operator. The procedure is much more delicate.

One thing that is of interest is to determine how far is the weakly convergent sequence from a strongly converging one. The simplest tool used for that are defect measures, the accumulation points of bounded L^1 sequences

$$|u_n - u|^p \overset{*}{\rightharpoonup} \nu \; .$$

This approach was studied by Ron DiPerna, Andrew Majda and Pierre-Louis Lions in the ∼1980.

Sketch of the Tartar programme $\sim$1980

Physical laws are often expressed as systems of partial differential equations, of which some equations can be nonlinear.

It turned out that it is useful to distinguish between two types of physical laws:

(linear) conservation laws ... mass, energy, momentum, charge etc.
These are generally valid physical laws.

(nonlinear) constitution laws ... elastic fluids, electrodynamics of continua
These laws characterise particular types of materials.

How to describe the interaction of nonlinear constitutive assumptions and linear conservation laws?

Example: electrostatics

D – electric induction, E – total electric field, ρ – charge density

Maxwell: $\operatorname{div} \mathsf{D} = \rho$, $\quad \operatorname{rot} \mathsf{E} = 0$
These are general conservation laws (system of linear pde-s)

A particular material is characterised by the relation: $\mathsf{D} = \mathsf{A}(\mathsf{E})$, where A is generally nonlinear.
In vacuum: $\mathsf{A}(\mathsf{E}) = \varepsilon_0 \mathsf{E}$, sometimes also linearised $\mathsf{A}(\mathsf{E}) = \mathbf{A}\mathsf{E}$, where matrix $\mathbf{A}$ depends on the space variable.

On a simply connected domain $\mathsf{E} = -\nabla u$ (a gradient of a potential), so by eliminating D from the system in general we get a nonlinear pde:

$$-\operatorname{div} (\mathsf{A}(\nabla u)) = \rho \, .$$

What can be said about nonlinear constraints?

Weak convergence is well behaved with respect to linear operators. However, we would like to consider nonlinear laws as well.

For simplicity, take L^∞ with weak $*$ topology and $F : \mathbf{R}^r \longrightarrow \mathbf{R}$ continuous (so that $F \circ u_n$ is again a bounded sequence, if u_n is such in L^∞).

Theorem. Let $K \subseteq \mathbf{R}^r$ be a bounded set, (u_n) a sequence in $\mathrm{L}^\infty(\Omega; K)$, $\mathsf{u}_n \xrightarrow{*} \mathsf{u}$.

Then $\mathsf{u}(\mathbf{x}) \in \mathsf{Cl}\,\mathrm{conv}K$ (a.e. $\mathbf{x}$).

Conversely, for $\mathsf{u} \in \mathrm{L}^\infty(\Omega; \mathsf{Cl}\,\mathrm{conv}K)$ there is a *sequence* $\mathsf{u}_n \in \mathrm{L}^\infty(\Omega; K)$ such that $\mathsf{u}_n \xrightarrow{*} \mathsf{u}$.

[If K is not bounded, the converse is not true.]

An example

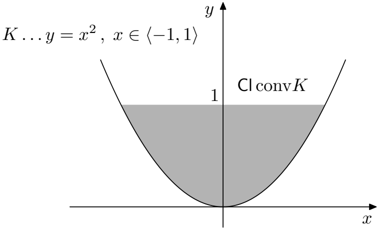
For the sequence $u_n(x) = \sin nx\,,\ x \in \Omega = \langle -\pi, \pi \rangle$ we have (in L^∞)

$$u_n \xrightarrow{*} 0$$
$$u_n^2 \xrightarrow{*} \frac{1}{2}$$

In general, if $u_n \xrightarrow{*} u$ then also $F \circ u_n \xrightarrow{*} F \circ u$ for a linear function F, but not necessarily for a nonlinear … We need Young measures for that.

Another approach is based on the previous theorem … $\mathsf{v}_n := \begin{bmatrix} u_n \\ u_n^2 \end{bmatrix}$

$$\mathsf{v}_n \in \mathrm{L}^\infty(\Omega; K)$$
$$\mathsf{v}_n \xrightarrow{*} ?$$



Theorem. Let (u_n) be a sequence in $\mathrm{L}^\infty(\Omega; K)$.
Then there is a subsequence (u_{n_k}) and a weakly $*$ measurable family of probability measures $(\nu_x, x \in \Omega)$ supported on $\mathsf{Cl}\, K$, such that for any continuous function F on $\mathsf{Cl}\, K$ one has

$$F \circ \mathsf{u}_{n_k} \xrightarrow{\;*\;} \langle \nu_\cdot, F \rangle = \int_{\mathsf{Cl}\, K} F(\lambda)\, d\nu_\cdot(\lambda)\,.$$

If K is bounded, the converse is also true.

(More precisely: $\nu \in \mathrm{L}^\infty_*(\Omega; \mathcal{M}_b(\mathsf{Cl}\, K))$, as $\mathcal{M}_b(\mathsf{Cl}\, K)$ is not reflexive.)

Young measures — an application

Let us see how the above can be applied in describing the limit of $F \circ \mathsf{u}_n$ (at least on a subsequence). In an earlier example:

$$\mathsf{E}_n \xrightarrow{*} \mathsf{E} \quad \Longrightarrow \quad \mathsf{D}_n \xrightarrow{*} \int \mathsf{A}(\lambda) d\nu_\cdot(\lambda) \ .$$

On a subsequence we get that

$$u_n \xrightarrow{*} \int \lambda d\nu_\cdot(\lambda) \ .$$

Conversely, if for any continuous F holds:

$$F \circ \mathsf{u}_n \xrightarrow{*} \int F(\lambda)\, d\nu_\cdot(\lambda) \,,$$

then necessarily $\nu_x = \delta_{u(x)}$, and the sequence converges strongly.

div − rot lemma: example in electrostatics

On the microscopic level the fields obey the Maxwell system: $\operatorname{div} \mathsf{D}_n = \rho$ and $\operatorname{rot} \mathsf{E}_n = 0$, and we have the electrostatic energy $\int \mathsf{E}_n \cdot \mathsf{D}_n$.

What can we say about that energy on the macroscopic scale?

$$\mathsf{E}_n \xrightarrow{\mathrm{L}^2} \mathsf{E} \qquad \text{and} \qquad \mathsf{D}_n \xrightarrow{\mathrm{L}^2} \mathsf{D}\ .$$

$$\mathsf{E}_n \cdot \mathsf{D}_n \xrightarrow{\mathcal{M}_b *} \mathsf{E} \cdot \mathsf{D}\ .$$

This is the consequence of the famous div-rot lemma (Murat, Tartar), and the physical meaning is that there is no hidden electrostatic energy.

Compactness by compensation

$\mathsf{u}_n \longrightarrow \mathsf{u}_0$ in $\mathrm{L}^2(\Omega;\mathbf{R}^r)$, $\mathcal{A}\nabla\mathsf{u}_n = \mathbf{A}^k\partial_k\mathsf{u}_n$ precompact in $\mathrm{H}^{-1}_{\mathrm{loc}}(\Omega;\mathbf{R}^r)$ ($\mathcal{A}$ is a third rank tensor, with constant coefficients).

A characteristic set:

$$\mathcal{V} := \left\{(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathbf{R}^r \times S^{d-1} : \mathcal{A}(\boldsymbol{\xi}\otimes\boldsymbol{\lambda}) = \mathbf{A}^k\boldsymbol{\lambda}\xi_k = 0\right\} ,$$

and its projection to the physical space:

$$\Lambda := \left\{\boldsymbol{\lambda} \in \mathbf{R}^r : (\exists\, \boldsymbol{\xi} \in S^{d-1})\ (\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathcal{V}\right\} .$$

Theorem. *For any quadratic form Q, for which $Q(\Lambda) \geqslant 0$, any weak $*$ accumulation point l of sequence $Q(\mathsf{u}_n)$ satisfies $l \geqslant Q(\mathsf{u}_0)$.* ∎

Example. $\mathsf{u}_n \longrightarrow \mathsf{u}_0$ in $\mathrm{L}^2(\mathbf{R}^2;\mathbf{R}^2)$, while $(\partial_1 u_n^1)$ and $(\partial_2 u_n^2)$ are bounded in $\mathrm{L}^2(\mathbf{R}^2)$ (therefore precompact in $\mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^2)$). The characteristic set is
$\mathcal{V} = \{(\boldsymbol{\lambda},\boldsymbol{\xi}) \in \mathbf{R}^2 \times S^1 : \xi_1\lambda^1 = \xi_2\lambda^2 = 0\}$, and its projection
$\Lambda = \{\boldsymbol{\lambda} \in \mathbf{R}^2 : \lambda^1\lambda^2 = 0\}$.
$Q(\boldsymbol{\lambda}) := \lambda^1\lambda^2$ annuls on Λ ($\pm Q(\Lambda) \geqslant 0$).
Therefore any accumulation point of $u_n^1 u_n^2$ is equal to $u_0^1 u_0^2$ (weak $*$ in measures).

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What are H-measures?

Mathematical objects introduced by:

- Luc Tartar, motivated by intended applications in homogenisation (H), and
- Patrick Gérard, whose motivation were certain problems in kinetic theory (and who called these objects *microlocal defect measures*).

Start from $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$, and take the Fourier transform:

$$\widehat{\varphi u_n}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} (\varphi u_n)(\mathbf{x}) d\mathbf{x} \ .$$

As φu_n is supported on a fixed compact set K, so $|\widehat{\varphi u_n}(\boldsymbol{\xi})| \leqslant C$.
Furthermore, $u_n \rightharpoonup 0$, and from the definition $\widehat{\varphi u_n}(\boldsymbol{\xi}) \longrightarrow 0$ pointwise.

By the Lebesgue dominated convergence theorem applied on bounded sets

$$\widehat{\varphi u_n} \longrightarrow 0 \text{ strong, i.e. strongly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d) \ .$$

On the other hand, by the Plancherel theorem: $\|\widehat{\varphi u_n}\|_{\mathrm{L}^2(\mathbf{R}^d)} = \|\varphi u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}$.

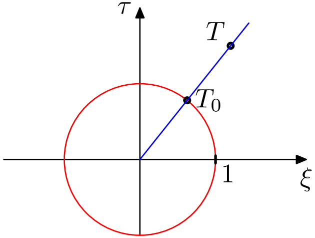
If $\varphi u_n \not\rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, then $\widehat{\varphi u_n} \not\rightharpoonup 0$; some information must go to infinity.

How does it go to infinity in various directions?
We can look along rays, or some other curves (like parabolas).

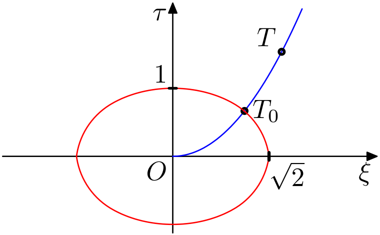
Rough geometric idea

Take a sequence $\mathrm{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^2)$, and integrate $|\widehat{\varphi \mathrm{u}_n}|^2$ along

rays and project onto S^1



parabolas and project onto P^1



In $\mathbf{R}^2$ we have a compact curve (a surface in higher dimensions):

$$S^1 \dots\ r^2(\tau,\xi) := \tau^2 + \xi^2 = 1 \qquad P^1 \dots\ \rho^2(\tau,\xi) := (\xi/2)^2 + \sqrt{(\xi/2)^4 + \tau^2} = 1$$

and projection of $\mathbf{R}^2_* = \mathbf{R}^2 \setminus \{0\}$ onto the curve (surface):

$$p(\tau,\xi) := \left(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\right) \qquad \pi(\tau,\xi) := \left(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\right)$$

Analytic picture

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.
The norm is again equal to $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Delicate part: a is given only on S^1 or P^1.
We extend it by the projections, p or π: if α is a function defined on a compact surface, we take $a := \alpha \circ p$ or $a := \alpha \circ \pi$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\Big) \qquad a(\tau, \xi) := \alpha\Big(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\Big)$$

The precise scaling is contained in the projections, not the surface.

The surface is chosen to be orthogonal to the curves we are projecting along, allowing for easier integration by parts.

Existence of H-measures

Theorem. If $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2{}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^r)$, then there exists its subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ onRadon measure $\boldsymbol{\mu}$ on distribution of order zero $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times S^{d-1} \qquad \mathbf{R}^d \times P^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and

$$\psi \in \mathrm{C}(S^{d-1}) \qquad \psi \in \mathrm{C}(P^{d-1})$$

one has

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ p\pi)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int\limits_{\mathbf{R}^d \times S^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi}) \qquad = \int\limits_{\mathbf{R}^d \times P^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi})\,.$$

There are some other variants: ultraparabolic, fractional, ...

First commutation lemma

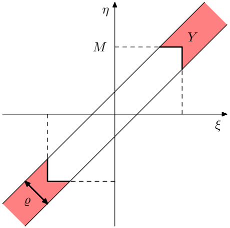
Lemma. (general form of the first commutation lemma — Luc Tartar)
If $b \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ *satisfy the condition*

$$(\forall\, \rho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})| \leqslant \varepsilon \;\; (\text{a.e. } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \rho)) \,,$$

then $C := [\mathcal{A}_a, M_b]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$.

For given $M, \rho \in \mathbf{R}^+$ denote the set

$$Y = Y(M, \rho) = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geqslant M \;\&\; |\boldsymbol{\xi} - \boldsymbol{\eta}| \leqslant \rho\} \,.$$



[older results by H. O. Cordes (JFA, 1975)]

The importance of First commutation lemma

If we take $\mathsf{u}_n = (u_n, v_n)$, and consider $\mu = \mu_{12}$, we have

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} \psi \, d\boldsymbol{\xi} &= \lim_{n'} \overline{\langle \mathcal{A}_\psi(\varphi_1 u_{n'}) | \varphi_2 v_{n'} \rangle} \\ &= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} \, d\mathbf{x} \\ &= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} \, d\mathbf{x} = \langle \mu, (\varphi_1 \bar\varphi_2) \boxtimes \psi \rangle \,.\end{aligned}$$

Thus the limit is a bilinear functional in $\varphi_1 \bar\varphi_2$ and ψ, and we have the bound:

$$\Big| \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} d\mathbf{x} \Big| \leqslant C \|\psi\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|\varphi_1 \overline{\varphi_2}\|_{\mathrm{C}_0(\mathbf{R}^d)} \,.$$

This bilinear functional can be related to a kernel distribution, which is positive. Thus, the distribution is in fact a Radon measure, giving the result.

Luc Tartar usually preferred to prove this result without referring to the Kernel theorem.

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Good bounds: the Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d) \,, \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

can be extended to a continuous mapping $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$.

Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2} \leqslant |\boldsymbol{\xi}| \leqslant r} |\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|} \,,$$

then for any $p \in \langle 1, \infty\rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty) \,.$$

∎

For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

Existence of H-distributions (main theorem)

Theorem. *If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$ *and* $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^q(\mathbf{R}^d)$ *for some* $q \geqslant \max\{p', 2\}$*, then there exist subsequences* $(u_{n'})$*,* $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$*, such that for every* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ *we have:*

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} &= \lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} \\ &= \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle \,,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$ *is the Fourier multiplier operator with symbol* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$*.* ∎

We call the functional μ the *H-distribution* corresponding to (a subsequence of) (u_n) and (v_n).

Of course, for $q \in \langle 1, \infty \rangle$ the weak $*$ convergence coincides with the weak convergence.

In fact, $\mu \in \mathcal{D}'_{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$.

Some remarks

If (u_n), (v_n) are defined on $\Omega \subseteq \mathbf{R}^d$, extension by zero to $\mathbf{R}^d$ preserves the convergence, and we can apply the Theorem. μ is supported on $\mathrm{Cl}\,\Omega \times \mathrm{S}^{d-1}$.

In Theorem we distinguish $u_n \in \mathrm{L}^p(\mathbf{R}^d)$ and $v_n \in \mathrm{L}^q(\mathbf{R}^d)$. For $p \geqslant 2$, $p' \leqslant 2$ and we can take $q \geqslant 2$; this covers the L^2 case (including $u_n = v_n$).
The assumptions of Theorem imply that $u_n, v_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, resulting in a distribution μ of order zero (a Radon measure, not necessary bounded), instead of a more general distribution.
The real improvement in Theorem is for $p < 2$.

For applications, of interest is to extend the result to vector-valued functions. For $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

In contrast to H-measures, we cannot consider H-distributions corresponding to the same sequence, but only to a pair of sequences, and H-distribution would correspond to non-diagonal blocks for H-measures.

First commutation lemma

$\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ satisfies the conditions of the Hörmander-Mihlin theorem. Therefore, $\mathcal{A}_\psi$ and B are bounded operators on $\mathrm{L}^r(\mathbf{R}^d)$, for any $r \in \langle 1, \infty \rangle$. We are interested in the properties of their commutator, $C = \mathcal{A}_\psi B - B\mathcal{A}_\psi$. If $p < r$, we can apply the classical interpolation inequality:

$$\|Cv_n\|_p \leqslant \|Cv_n\|_2^\alpha \|Cv_n\|_r^{1-\alpha} \ ,$$

for $\alpha \in \langle 0, 1 \rangle$ such that $1/p = \alpha/2 + (1-\alpha)/r$.
As C is compact on $\mathrm{L}^2(\mathbf{R}^d)$ by Tartar's First commutation lemma, while it is bounded on $\mathrm{L}^r(\mathbf{R}^d)$, we get the claim.

For the most interesting case, where $p = r$, we need a better result: the Krasnosel'skij theorem (in fact, its extension to unbounded domains [N.A., M. Mišur, D. Mitrović (2018)]).

Lemma. *Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty \rangle \setminus \{2\}$. Then A is also compact on any $\mathrm{L}^p(\mathbf{R}^d)$, where $1/p = \theta/2 + (1-\theta)/r$, for a $\theta \in \langle 0, 1 \rangle$.* ∎

Therefore, the commutator C is compact on all $\mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1, \infty \rangle$.

A particular Nemyckij operator

Canonical choice of $\mathrm{L}^{p'}$ sequence corresponding to an L^p, $p \in \langle 1, \infty \rangle$, sequence (u_n) is given by $v_n = \Phi_p(u_n)$, where Φ_p is an operator from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\Phi_p(u) = |u|^{p-2}u$.

Φ_p is a nonlinear Nemytskij operator, continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ and additionally we have the following bound

$$\|\Phi_p(u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} \leqslant \|u\|^{p/p'}_{\mathrm{L}^p(\mathbf{R}^d)}.$$

It maps bounded sets in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ topology to bounded sets in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ topology. Hence for an L^p bounded sequence (u_n), we get that $(\Phi_p(u_n))$ is weakly precompact in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

It is continuous from $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

Example: concentration

$u \in \mathrm{L}^p_c(\mathbf{R}^d)$, and define $u_n(\mathbf{x}) = n^{\frac{d}{p}} u(n(\mathbf{x}-\mathbf{z}))$ for some $\mathbf{z} \in \mathbf{R}^d$.

Simple change of variables: $\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)} = \|u\|_{\mathrm{L}^p(\mathbf{R}^d)}$ and $u_n \rightharpoonup 0$ in $\mathrm{L}^p(\mathbf{R}^d)$.

Indeed, the sequence is bounded, while for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$

$$\begin{aligned}\int_{\mathbf{R}^d} u_n(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} &= \int_{\mathbf{R}^d} n^{d/p} u(n(\mathbf{x}-\mathbf{z}))\varphi(\mathbf{x})d\mathbf{x} \\ &= \int_{\mathbf{R}^d} n^{d/p-d} u(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\ &= \frac{1}{n^{d/p'}} \int_{\mathbf{R}^d} u(\mathbf{y})\chi_{\operatorname{supp} u}(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\ &\leqslant \left(\frac{\mathrm{vol}(\operatorname{supp} u)}{n^d}\right)^{1/p'} \|u\|_{\mathrm{L}^p(\mathbf{R}^d)} \max_{\mathbf{R}^d} |\varphi|.\end{aligned}$$

Passing to the limit, we get our claim.

The H-distribution corresponding to sequences (u_n) and $(\Phi_p(u_n))$ is given by $\delta_{\mathbf{z}} \boxtimes \nu$, where ν is a distribution on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$\langle \nu, \psi \rangle = \int_{\mathbf{R}^d} u(\mathbf{x}) \overline{\mathcal{A}_{\bar\psi}(|u|^{p-2}u)(\mathbf{x})} d\mathbf{x}.$$

This distribution is not a Radon measure.

Localisation principle

Theorem. *Take* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $f_n \to 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$, *for some* $q \in \langle 1, d \rangle$, *such that*

$$\mathsf{div}\,(\mathsf{a}(\mathbf{x})u_n(\mathbf{x})) = f_n(\mathbf{x}) \ .$$

Take an arbitrary (v_n) *bounded in* $\mathrm{L}^\infty(\mathbf{R}^d)$, *and by* μ *denote the* H*-distribution corresponding to a subsequence of* (u_n) *and* (v_n). *Then*

$$(\mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi})\mu(\mathbf{x}, \boldsymbol{\xi}) = 0$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$, $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}$ *being the symbol of the linear PDO with* C_0^κ *coefficients.* ∎

In order to prove the theorem, we need a particular multiplier, the so called (Marcel) Riesz potential $I_1 := \mathcal{A}_{|2\pi\boldsymbol{\xi}|^{-1}}$, and the Riesz transforms $R_j := \mathcal{A}_{\frac{\xi_j}{i|\boldsymbol{\xi}|}}$.

Note that

$$\int I_1(\phi)\partial_j g = \int (R_j\phi)g, \quad g \in \mathcal{S}(\mathbf{R}^d).$$

Using the density argument and that R_j is bounded on $\mathrm{L}^p(\mathbf{R}^d)$, we conclude $\partial_j I_1(\phi) = -R_j(\phi)$, for $\phi \in \mathrm{L}^p(\mathbf{R}^d)$.

Compactness by compensation: L^2 case

It is well known that weak convergences are ill behaved under nonlinear transformations. Only in some particular cases of compensation it is even possible to pass to the limit in a product of two weakly converging sequences.

The prototype of this compensation effect is Murat-Tartar's div-rot lemma.

For simplicity consider 2D case, (u_n^1, u_n^2) and (v_n^1, v_n^2) converging to zero weakly in $\mathrm{L}^2(\mathbf{R}^2)$, such that $(\partial_x u_n^1 + \partial_y u_n^2)$ and $(\partial_y v_n^1 - \partial_x v_n^2)$ are both contained in a compact set of $\mathrm{H}^{-1}_{loc}(\mathbf{R}^2)$ (which then implies that they converge to zero strongly in $\mathrm{H}^{-1}_{loc}(\mathbf{R}^2)$).

We can define $\mathsf{U}_n := \begin{bmatrix} \mathsf{u}_n \\ \mathsf{v}_n \end{bmatrix}$, which (on a subsequence) defines a 4×4 H-measure $\boldsymbol{\mu}$. By the localisation principle, as the above relations can be written in the form ($\mathbf{A}^1, \mathbf{A}^2$ are 4×4 constant matrices with all entries zero except $A^1_{11} = A^2_{12} = A^2_{33} = 1$ and $A^1_{34} = -1$)

$$\mathbf{A}^1 \partial_1 \mathsf{U}_n + \mathbf{A}^2 \partial_2 \mathsf{U}_n \to 0 \text{ strongly in } \mathrm{H}^{-1}_{loc}(\mathbf{R}^2)^4 ,$$

the corresponding H-measure satisfies $(\xi_1 \mathbf{A}^1 + \xi_2 \mathbf{A}^2)\boldsymbol{\mu} = \mathbf{0}$. After straightforward calculations this shows that $u_n^1 v_n^1 + u_n^2 v_n^2 \longrightarrow 0$ weak $*$ in the sense of Radon measures (and therefore in the sense of distributions as well).

What for sequences in L^p?

For the above we have used only the non-diagonal blocks $\boldsymbol{\mu}_{12} = \boldsymbol{\mu}_{21}^*$ of

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix} ,$$

corresponding to products of u_n^i and v_n^j; in fact, the calculation shows that $\mu_{12}^{11} + \mu_{12}^{22} = 0$, which gives the above result.

Assume now (u_n^1, u_n^2) and (v_n^1, v_n^2) converging to zero weakly in $\mathrm{L}^p(\mathbf{R}^2)$ and $\mathrm{L}^{p'}(\mathbf{R}^2)$, and $(\partial_1 u_n^1 + \partial_2 u_n^2)$ bounded in $\mathrm{L}^p(\mathbf{R}^2)$, while $(\partial_2 v_n^1 - \partial_1 v_n^2)$ in $\mathrm{L}^{p'}(\mathbf{R}^2)$ (thus precompact in $\mathrm{W}_{loc}^{-1,p}(\mathbf{R}^2)$, and $\mathrm{W}_{loc}^{-1,p'}(\mathbf{R}^2)$).

Then $(u_n^1 v_n^1 + u_n^2 v_n^2)$ is bounded in $\mathrm{L}^1(\mathbf{R}^2)$, so also in $\mathcal{M}_b$ (Radon measures), and by weak $*$ compactness it has a weakly converging subsequence. However, we can say more—the whole sequence converges to zero.

Denote by μ^{ij} the H-distribution corresponding to (some sub)sequences (of) (u_n^1, u_n^2) and (v_n^1, v_n^2).
Since $(\partial_1 u_n^1 + \partial_2 u_n^2)$ is bounded in $\mathrm{L}^p(\mathbf{R}^2)$, and $(\partial_2 v_n^1 - \partial_1 v_n^2)$ is bounded in $\mathrm{L}^{p'}(\mathbf{R}^2)$, they are weakly precompact, while the only possible limit is zero, so

$$\partial_1 u_n^1 + \partial_2 u_n^2 \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^p , \qquad \text{and}$$
$$\partial_2 v_n^1 - \partial_1 v_n^2 \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^{p'} .$$

From the compactness of the Riesz potential I_1 mentioned above, we conclude that for $\varphi \in \mathrm{C}_c(\mathbf{R}^2)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ the following limit holds in $\mathrm{L}^p(\mathbf{R}^2)$:

$$\mathcal{A}_{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\frac{\xi_1}{|\boldsymbol{\xi}|}}(\varphi u_n^1) + \mathcal{A}_{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\frac{\xi_2}{|\boldsymbol{\xi}|}}(\varphi u_n^2) = \mathcal{A}_{\frac{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}(\partial_1(\varphi u_n^1) + \partial_2(\varphi u_n^2)) \to 0\,.$$

Multiplying it first by φv_n^1 and then by φv_n^2, integrating over $\mathbf{R}^2$ and passing to the limit, we conclude from the existence theorem that:

$$\xi_1 \mu^{11} + \xi_2 \mu^{21} = 0, \qquad \text{and} \qquad \xi_1 \mu^{12} + \xi_2 \mu^{22} = 0\,.$$

Next, take

$$w_n^j = \varphi \mathcal{A}_{\frac{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}(\varphi u_n^j) \in \mathrm{W}^{1,p'}(\mathbf{R}^d), \quad j = 1, 2.$$

From the last limits on the preceeding slide we get

$$\langle (\varphi v_n^1, -\varphi v_n^2), \nabla w_n^j \rangle = -\langle \mathsf{rot}\,(\varphi v_n^1, \varphi v_n^2), w_n^j \rangle \to 0 \ \text{ as } \ n \to \infty,$$

for $j = 1, 2$. Rewriting it in the integral formulation, we obtain again from the existence theorem:

$$\xi_2 \mu^{11} - \xi_1 \mu^{12} = 0, \quad \xi_2 \mu^{21} - \xi_1 \mu^{22} = 0.$$

From the algebraic relations above, we can easily conclude

$$\xi_1 \left(\mu^{11} + \mu^{22}\right) = 0 \ \text{ and } \ \xi_2 \left(\mu^{11} + \mu^{22}\right) = 0,$$

implying that the distribution $\mu^{11} + \mu^{22}$ is supported on the set $\{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap \mathrm{S}^1 = \emptyset$, which implies $\mu^{11} + \mu^{22} \equiv 0$.
After inserting $\psi \equiv 1$ in the definition of H-distribution, we immediately reach the conclusion.

This proof is similar to the L^2 case, but it should be noted that we had used only a non-diagonal block of 4×4 H-measure, which corresponds to the only available 2×2 H-distribution.

There is no reason to limit oneself to two dimensions; take (u_n) and (v_n) converging weakly to zero in $\mathrm{L}^p(\mathbf{R}^d)^d$ and $\mathrm{L}^{p'}(\mathbf{R}^d)^d$, and by $\boldsymbol{\mu}$ denote $d \times d$ matrix H-distribution corresponding to some chosen subsequences of (u_n) and (v_n).

Theorem. *Let (u_n) and (v_n) be vector valued sequences converging to zero weakly in $\mathrm{L}^p(\mathbf{R}^d)^d$ and $\mathrm{L}^{p'}(\mathbf{R}^d)^d$, respectively. Assume the sequence $(\operatorname{div} \mathsf{u}_n)$ is bounded in $\mathrm{L}^p(\mathbf{R}^d)$, and the sequence $(\operatorname{rot} \mathsf{v}_n)$ is bounded in $\mathrm{L}^{p'}(\mathbf{R}^d)^{d\times d}$. Then the sequence $(\mathsf{u}_n \cdot \mathsf{v}_n)$ converges to zero in the sense of distributions (or vaguely in the sense of Radon measures).* ∎

The results carry on to loc spaces as well.

Semiclassical measures

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega; \mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exists a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}_{\rm b}(\Omega \times \mathbf{R}^d; \mathrm{M_r}(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big(\widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi})\Big) \psi(\omega_{n'}\boldsymbol{\xi})\, d\boldsymbol{\xi} = \Big\langle \boldsymbol{\mu}_{sc}^{(\omega_n)}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \Big\rangle .$$

Measure $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ *we call the semiclassical measure with characteristic length* (ω_n) *corresponding to the (sub)sequence* $(\mathsf{u}_{n'})$.

Definition (u_n) is (ω_n)-oscillatory if
$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \quad \lim_{R\to\infty} \limsup_n \int_{|\boldsymbol{\xi}| \geqslant \frac{R}{\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} = 0\,.$

Theorem.

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\rm loc}} 0 \iff \boldsymbol{\mu}_{sc}^{(\omega_n)} = \mathbf{0} \quad \& \quad (\mathsf{u}_n) \textit{ is } (\omega_n)-\textit{oscillatory}\,.$$

Localisation principle for semiclassical measures

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\mathbf{P}_n \mathsf{u}_n := \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

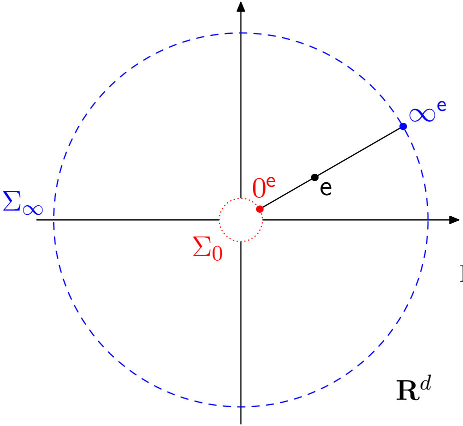
Then we have

$$\mathbf{p} \boldsymbol{\mu}_{sc}^{\top} = \mathbf{0} \,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

Problem: $\boldsymbol{\mu}_{sc} = \mathbf{0}$ is not enough for the strong convergence!

Compatification of $\mathbf{R}^d \setminus \{0\}$



$$\Sigma_0 := \{0^{\mathsf{e}} \;:\; \mathsf{e} \in \mathrm{S}^{d-1}\}$$

$$\Sigma_\infty := \{\infty^{\mathsf{e}} \;:\; \mathsf{e} \in \mathrm{S}^{d-1}\}$$

$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

Corollary. *a)* $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \boldsymbol{\pi} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$. ∎

One-scale H-measures

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega; \mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exists a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}_{\mathrm{b}}(\Omega \times \mathbf{R}^d; \mathrm{M}_r(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big((\widehat{\varphi_1 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \otimes (\widehat{\varphi_2 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \Big) \psi(\omega_{n'} \boldsymbol{\xi}) \, d\boldsymbol{\xi} = \Big\langle \boldsymbol{\mu}_{sc}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \Big\rangle \, .$$

Measure $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ *is called the semiclassical measure with characteristic length* (ω_n) *corresponding to the (sub)sequence* $(\mathsf{u}_{n'})$. ∎

Luc Tartar: *The general theory of homogenization: A personalized introduction*, *Springer*, 2009.

Luc Tartar: *Multi-scale H-measures*, *Discrete and Continuous Dynamical Systems, S* **8** (2015) 77–90.

N. A., Marko Erceg, Martin Lazar: *Localisation principle for one-scale H-measures*, *Journal of Functional Analysis* **272** (2017) 3410–3454.

Marko Erceg, Martin Lazar: *Characteristic scales of bounded* L^2 *sequences*, *Asymptotic Analysis* **109** (2018) 171–192.

Idea of the proof

Tartar's approach:

- $\mathsf{v}_n(\mathbf{x}, x^{d+1}) := \mathsf{u}_n(\mathbf{x}) e^{\frac{2\pi i x^{d+1}}{\omega_n}} \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega \times \mathbf{R}; \mathbf{C}^r)$
- $\boldsymbol{\nu}_H \in \mathcal{M}(\Omega \times \mathbf{R} \times \mathrm{S}^d; \mathrm{M_r}(\mathbf{C}))$
- $\boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}$ is obtained from $\boldsymbol{\nu}_H$ (suitable projection in x^{d+1} and ξ_{d+1})

Our approach:

- First commutation lemma:

Lemma. *Let $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, and denote $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. Then the commutator can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K \,,$$

where K is a compact operator on $\mathrm{L}^2(\mathbf{R}^d)$, while $\tilde{C}_n \longrightarrow 0$ in the operator norm on $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$. ∎

- standard procedure: (a variant of) the kernel theorem, separability, . . .

Some properties of $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$

Theorem.

a) $\boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$, $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \geqslant \mathbf{0}$

b) $\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}$

c) $\mathrm{tr}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 \iff (\mathsf{u}_n)$ *is* (ω_n) *– oscillatory*

■

Theorem. $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathrm{C}_0(\mathbf{R}^d)$, $\tilde\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\omega_n \to 0^+$,

a) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle = \langle \boldsymbol{\mu}^{(\omega_n)}_{sc}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle$,

b) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi \circ \boldsymbol{\pi}\rangle = \langle \boldsymbol{\mu}_H, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi\rangle$,

where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

■

Localisation principle

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

where

- $l \in 0..m$
- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

Lemma. *a)* $(\mathrm{C}(\varepsilon_n))$ *is equivalent to*

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + |\boldsymbol{\xi}|^l + \varepsilon_n^{m-l} |\boldsymbol{\xi}|^m} \longrightarrow 0 \quad \textit{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)\,.$$

b) $(\exists\, k \in l..m)$ $\mathsf{f}_n \longrightarrow 0$ *in* $\mathrm{H}^{-k}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \quad \Longrightarrow \quad (\varepsilon_n^{k-l} \mathsf{f}_n)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$. ∎

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem. [Tartar (2009)] *Under previous assumptions and $l=1$, one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) corresponding to (u_n) satisfies*

$$\operatorname{supp}(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

Theorem. [N.A., Erceg, Lazar (2017)] *Under previous assumptions, one-scale H-measure $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ with characteristic length (ε_n) corresponding to (u_n) satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Localisation principle - final generalisation

Theorem. *Take* $\varepsilon_n > 0$ *bounded,* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *and*

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$*,* $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ *uniformly on compact sets, and* $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$*.*
Then for $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0, \infty]$*, the corresponding one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = 0 \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = \infty \end{cases}$$

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$*,* $n \in \mathbf{N}$*, we can take*

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

As a corollary from the previous theorem we can derive localisation principles for H-measures and semiclassical measures.

$\Omega \subseteq \mathbf{R}^d$ open, $p \in \langle 1, \infty \rangle$, $\frac{1}{p} + \frac{1}{p'} = 1$

Teorem

If $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ and $\omega_n \to 0^+$, then there exist $(u_{n'})$, $(v_{n'})$ and $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 v_{n'}}(\boldsymbol{\xi})} \psi(\omega_{n'} \boldsymbol{\xi}) \, d\boldsymbol{\xi} = \langle \mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \, .$$

The measure $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ is called the one-scale H-measure with characteristic length $(\omega_{n'})$ associated to the (sub)sequences $(u_{n'})$ and $(v_{n'})$.

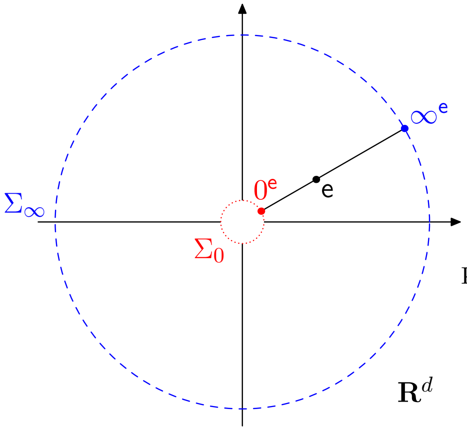
$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

Determine E such that

— $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is continuous

— The First commutation lemma is valid

Smooth compactification of $\mathbf{R}^d_*$



$$\Sigma_0 := \{0^{\mathsf{e}} \ : \ \mathsf{e} \in \mathrm{S}^{d-1}\}$$

$$\Sigma_\infty := \{\infty^{\mathsf{e}} \ : \ \mathsf{e} \in \mathrm{S}^{d-1}\}$$

$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d_* \cup \Sigma_0 \cup \Sigma_\infty$$

$\mathcal{T}$ radial translation for r_0

$$\mathbf{R}^d_* \ni \boldsymbol{\xi} \overset{\mathcal{T}}{\longmapsto} \frac{|\boldsymbol{\xi}| + r_0}{|\boldsymbol{\xi}|}\boldsymbol{\xi} \in \mathbf{R}^d \setminus \mathrm{K}[0, r_0] \ .$$

Modified stereographic projection $\mathcal{R}$

Denote $\mathrm{S}^d_I := \Big\{(\zeta_0, \boldsymbol{\zeta}) \in \mathrm{S}^d : \zeta_0 \in I\Big\}$, $I \subseteq [-1, 1]$.
Identify $\mathbf{R}^d$ with hyperplane $\xi_0 = 1$ in $\mathbf{R}^{1+d}$, and project it to the open upper unit hemisphere $\mathrm{S}^d_{\langle 0,1]}$; simple calculation gives us

$$\mathcal{R} : \mathbf{R}^d \to \mathrm{S}^d_{\langle 0,1]} \quad , \qquad \mathcal{R}(\boldsymbol{\xi}) = \left(\frac{1}{\sqrt{1+|\boldsymbol{\xi}|^2}}, \frac{\boldsymbol{\xi}}{\sqrt{1+|\boldsymbol{\xi}|^2}}\right).$$

Compactification: $\quad \mathcal{J} := \mathcal{R} \circ \mathcal{T}$
Since $\mathcal{R}(\mathbf{R}^d \setminus \mathrm{K}[0, r_0]) = \mathrm{S}^d_{\langle 0,1]}$, where $r_1 := (1 + r_0^2)^{-1/2}$, we have

$$\mathcal{J} : \mathbf{R}^d_* \to \mathrm{S}^d_{\langle 0,1]} \quad , \qquad \mathcal{J}(\boldsymbol{\xi}) = \left(\frac{1}{\sqrt{1+(|\boldsymbol{\xi}|+r_0)^2}}, \frac{|\boldsymbol{\xi}|+r_0}{\sqrt{1+(|\boldsymbol{\xi}|+r_0)^2}}\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right).$$

$\mathcal{J}$ is a C^∞-dipheomorphism, its inverse $\mathcal{J}^{-1} : \mathrm{S}^d_{\langle 0,1]} \to \mathbf{R}^d_*$ being

$$\mathcal{J}^{-1}(\zeta_0, \boldsymbol{\zeta}) = \frac{\boldsymbol{\zeta}}{\zeta_0} - r_0 \frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}.$$

$(\mathrm{S}^d_{[0,r_1]}, \mathcal{J})$ is a compactification of $\mathbf{R}^d_*$ $\qquad$ (as $\mathrm{S}^d_{[0,r_1]} = \mathrm{Cl}\,\mathrm{S}^d_{\langle 0,1]}$).

Extension of $\mathcal{J}$

It remains to relate $\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d_* \cup \Sigma_0 \cup \Sigma_\infty$ and $\mathrm{S}^d_{[0,r_1]}$.
Since

$$[0,\infty\rangle \ni x \mapsto \frac{1}{\sqrt{1+(x+r_0)^2}}$$

is strictly decreasing, for any sequence $(\boldsymbol{\xi}_n)$ in $\mathbf{R}^d_*$ we have

$$\lim_n \Big| \mathcal{J}(\boldsymbol{\xi}_n) - \Big(r_1, r_0 r_1 \frac{\boldsymbol{\xi}_n}{|\boldsymbol{\xi}_n|}\Big)\Big| = 0 \quad \Longleftrightarrow \quad \lim_n |\boldsymbol{\xi}_n| = 0 \ ,$$

$$\lim_n \Big| \mathcal{J}(\boldsymbol{\xi}_n) - \Big(0, \frac{\boldsymbol{\xi}_n}{|\boldsymbol{\xi}_n|}\Big)\Big| = 0 \quad \Longleftrightarrow \quad \lim_n |\boldsymbol{\xi}_n| = +\infty \ .$$

It is thus natural to extend $\mathcal{J}$ to $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ (hence also $\mathcal{J}^{-1}$ to $\mathrm{S}^d_{[0,r_1]}$) by

$$\mathcal{J}(0^{\mathsf{e}}) := (r_1, r_0 r_1 \mathsf{e}) \ , \qquad \mathcal{J}(\Sigma_0) = \mathrm{S}^d_{r_1}$$

$$\mathcal{J}(\infty^{\mathsf{e}}) := (0, \mathsf{e}) \ , \qquad \mathcal{J}(\Sigma_\infty) = \mathrm{S}^d_0 \ .$$

[N.B.the sphere at infinity Σ_∞ is mapped onto S^d_0]

Smooth test functions

By pulling back the Euclidean metric from $\mathrm{S}^d_{[0,r_1]}$ we can get topology on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$, thus defining $\mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

Of course, this can be extended for $\kappa \in \mathbf{N}_0 \cup \{\infty\}$

$$\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) := \Big\{\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) : \psi^* := (\mathcal{J}^{-1})^*\psi = \psi \circ \mathcal{J}^{-1} \in \mathrm{C}^\kappa(\mathrm{S}^d_{[0,r_1]})\Big\}$$

For $\kappa \in \mathbf{N}_0$ they are separable Banach algebras (as $\mathrm{C}^\kappa(\mathrm{S}^d_{[0,r_1]})$ are), with the norm $\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))} := \|\psi^*\|_{\mathrm{C}^\kappa(\mathrm{S}^d_{[0,r_1]})}$.

For $\kappa = \infty$, the sequence of norms $\big(\|\cdot\|_{\mathrm{C}^n(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}\big)_n$ makes $\mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ into a Fréchet space.

Clearly, the restriction of these functions to $\mathbf{R}^d_*$ is of the same class. Is the converse also true?

If such a continuous extension exists, it is unique, so we can identify each function in $\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ with one and only one in $\mathrm{C}^\kappa(\mathbf{R}^d_*)$ (embedding). The question is how to recognise the image of that embedding within $\mathrm{C}^\kappa(\mathbf{R}^d_*)$.

A criterion

By identifying a neighbourhood of Σ_0 with the product $[0,1\rangle \times \mathrm{S}^{d-1}$, using $\mathcal{J}_0(\boldsymbol{\xi}) := (|\boldsymbol{\xi}|, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})$, and analogously $\mathcal{J}_\infty(\boldsymbol{\xi}) := (\frac{1}{|\boldsymbol{\xi}|}, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|})$ for Σ_∞, one gets

Lemma. For any $\kappa \in \mathbf{N}_0 \cup \{\infty\}$, it is equivalent:

a) $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

b) $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d_*)$ and there exist $\tilde\psi_0, \tilde\psi_\infty \in \mathrm{C}^\kappa([0,1\rangle \times \mathrm{S}^{d-1})$ such that

$$\psi(\boldsymbol{\xi}) = \tilde\psi_0\Big(|\boldsymbol{\xi}|, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\ , \quad 0 < |\boldsymbol{\xi}| < 1$$

$$\psi(\boldsymbol{\xi}) = \tilde\psi_\infty\Big(\frac{1}{|\boldsymbol{\xi}|}, \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)\ , \quad |\boldsymbol{\xi}| > 1\ .$$

Corollary 1. Let $\psi \in \mathrm{C}^\kappa(\mathbf{R}^d)$ and let $\tilde\psi_\infty \in \mathrm{C}^\kappa([0,1\rangle \times \mathrm{S}^{d-1})$ be such that the last condition in Lemma holds. Then $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

Corollary 2. Let $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$. Then there exist unique functions $\psi_0, \psi_\infty \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ such that

$$\psi(\boldsymbol{\xi}) - \psi_0\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \longrightarrow 0\ , \quad |\boldsymbol{\xi}| \to 0\ ,$$

$$\psi(\boldsymbol{\xi}) - \psi_\infty\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big) \longrightarrow 0\ , \quad |\boldsymbol{\xi}| \to \infty\ .$$

If for $\psi \in \mathrm{C}(\mathbf{R}^d_*)$ there exist $\psi_0, \psi_\infty \in \mathrm{C}(\mathrm{S}^{d-1})$ such that the above holds, then $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

Estimates on the norms and examples of functions

By using the *generalised chain rule (Faá di Bruno) formula*, we get

Lemma For any $\kappa \in \mathbf{N}_0$ there are $c_\kappa, C_\kappa > 0$ such that for any $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$
\begin{aligned}
&c_\kappa \max\Big\{\|\tilde\psi_0\|_{\mathrm{C}^\kappa([0,\frac12]\times\mathrm{S}^{d-1})}, \|\psi\|_{\mathrm{C}^\kappa(\{\boldsymbol\xi\in\mathbf{R}^d:\frac14\leqslant|\boldsymbol\xi|\leqslant4\})}, \|\tilde\psi_\infty\|_{\mathrm{C}^\kappa([0,2]\times\mathrm{S}^{d-1})}\Big\} \\
&\qquad\leqslant \|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))} \\
&\qquad\leqslant C_\kappa \max\Big\{\|\tilde\psi_0\|_{\mathrm{C}^\kappa([0,\frac12]\times\mathrm{S}^{d-1})}, \|\psi\|_{\mathrm{C}^\kappa(\{\boldsymbol\xi\in\mathbf{R}^d:\frac14\leqslant|\boldsymbol\xi|\leqslant4\})}, \|\tilde\psi_\infty\|_{\mathrm{C}^\kappa([0,2]\times\mathrm{S}^{d-1})}\Big\} ,
\end{aligned}
$$

where functions $\tilde\psi_0, \tilde\psi_\infty$ are given in previous Lemma.

Corollary. Let $\boldsymbol\pi(\boldsymbol\xi) := \frac{\boldsymbol\xi}{|\boldsymbol\xi|}$ be the projection on $\mathbf{R}^d_*$ along rays to the unit sphere S^{d-1}.

a) $\big\{\psi\circ\boldsymbol\pi : \psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})\big\} \subseteq \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\kappa \in \mathbf{N}_0 \cup \{\infty\}$.

b) $\mathcal{S}(\mathbf{R}^d) \subseteq \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

c) $(\forall m, l \in \mathbf{N}_0\,,\ l \leqslant m)(\forall \boldsymbol\alpha \in \mathbf{N}_0^d\,,\ l \leqslant |\boldsymbol\alpha| \leqslant m)$
$\boldsymbol\xi \mapsto \frac{\boldsymbol\xi^{\boldsymbol\alpha}}{|\boldsymbol\xi|^l + |\boldsymbol\xi|^m} \in \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$. Ž

d) $(\forall m \in \mathbf{N})\ \ \boldsymbol\xi \mapsto \frac{1+|\boldsymbol\xi|^m}{(1+|\boldsymbol\xi|^2)^{\frac m2}}\,,\ \boldsymbol\xi \mapsto \frac{(1+|\boldsymbol\xi|^2)^{\frac m2}}{1+|\boldsymbol\xi|^m} \in \mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.

(Hörmander-)Mihlin theorem
If for $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ there exists $C > 0$ such that ($\kappa = [\frac{d}{2}] + 1$)

$$(\forall\, \boldsymbol{\xi} \in \mathbf{R}^d_*)(\forall\, \boldsymbol{\alpha} \in \mathbf{N}^d_0) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \quad \Longrightarrow \quad |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})| \leqslant \frac{C}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,,$$

then ψ is a Fourier multiplier for any $p \in \langle 1, \infty\rangle$. Moreover, we have

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_d \max\Bigl\{p, \frac{1}{p-1}\Bigr\} C\,.$$

Theorem. Any $\psi \in \mathrm{C}^{[\frac{d}{2}]+1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ satisfies Mihlin's condition; it holds

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_{d,p} C_d \|\psi\|_{\mathrm{C}^{[\frac{d}{2}]+1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}\,,$$

where $C_{d,p}$ is the constant from Mihlin's theorem, while C_d depends only on d.

Thus the linear mapping $\mathrm{C}^{[\frac{d}{2}]+1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) \ni \psi \mapsto \mathcal{A}_\psi \in \mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))$ is continuous.

Commutation lemma

multiplication by $\varphi \in \mathrm{L}^\infty(\mathbf{R}^d)$: $B_\varphi \mathsf{u} := \varphi \mathsf{u}$ bounded on $\mathrm{L}^p(\mathbf{R}^d)$, $p \in [0, \infty]$
Fourier multiplier of $\psi \in \mathrm{C}^{[\frac{d}{2}]+1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$: $\mathcal{A}_\psi \mathsf{u} := \bar{\mathcal{F}}(\psi \hat{\mathsf{u}})$
Taking $\omega_n \to 0^+$, and $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$, the sequence of commutators

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] := B_\varphi \mathcal{A}_{\psi_n} - \mathcal{A}_{\psi_n} B_\varphi$$

is bounded in $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))$, for any $p \in \langle 1, \infty \rangle$.

Lemma. For $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$ and assumptions as above, $C_n = \tilde{C}_n + K$, where K is a compact operator, while $\tilde{C}_n \longrightarrow 0$ in the operator norm on $\mathrm{L}^p(\mathbf{R}^d)$.

If $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ satisfies Mihlin's condition with C, then for any $a > 0$ the same is true also for $\psi_a := \psi(a\cdot)$ (the same constant C!).

Anisotropic distributions on manifolds without boundary

For simplicity, let $\Omega \subseteq \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^r_{\mathbf{y}}$ be open.
The general case on differentiable manifolds without boundary X and Y then easily follows using the local nature of distributions and the fact that every differentiable manifold is locally diffeomorphic to some Euclidean space.

For $l, m \in \mathbf{N}_0 \cup \{\infty\}$ consider $\mathrm{C}^{l,m}(\Omega)$

$$\left\{ f : \Omega \to \mathbf{C} : (\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d)(\forall \boldsymbol{\beta} \in \mathbf{N}_0^r)\ |\boldsymbol{\alpha}| \leqslant l, |\boldsymbol{\beta}| \leqslant m \implies \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega) \right\} .$$

K_n nested compacts in $\Omega = \bigcup_{n \in \mathbf{N}} K_n$; define (sequence for either $l, m = \infty$)

$$p^{l,m}_{K_n}(f) := \max_{|\boldsymbol{\alpha}| \leqslant l, |\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} .$$

For $l, m \in \mathbf{N}_0 \cup \{\infty\}$ these seminorms turn $\mathrm{C}^{l,m}(\Omega)$ into a separable Fréchet space with the topology of uniform convergence on compact sets of functions and their derivatives up to order l in $\mathbf{x}$ and m in $\mathbf{y}$, while $\mathrm{C}^\infty_c(\Omega)$ is dense in it.
For a compact set $K \subseteq \Omega$ and finite l and m, its subspace

$$\mathrm{C}^{l,m}_K(\Omega) := \left\{ f \in \mathrm{C}^{l,m}(\Omega) :\ \mathrm{supp}\, f \subseteq K \right\}$$

is a Banach space, and its inherited topology from $\mathrm{C}^{l,m}(\Omega)$ is a norm topology determined by

$$\|f\|_{l,m,K} := p^{l,m}_K(f) .$$

Anisotropic distributions ... (cont.)

If $l = \infty$ or $m = \infty$, we shall not get a Banach space, but a Fréchet space. Finally, the set of all $\mathrm{C}^{l,m}(\Omega)$ functions with compact support

$$\mathrm{C}_c^{l,m}(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}_{K_n}^{l,m}(\Omega) \,,$$

we equip with the topology of strict inductive limit, obtaining a complete topological space.

Any continuous linear functional on $\mathrm{C}_c^{l,m}(\Omega)$ we call a *distribution of anisotropic order*, and such functionals form a vector space $\mathcal{D}'_{l,m}(\Omega) := (\mathrm{C}_c^{l,m}(\Omega))'$.

Since $\Omega \subseteq \mathbf{R}^d$ is open and $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ is compact (hence closed), we can interpret $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$ as a smooth manifold with boundary. Again, it is enough to define distributions on $\Omega \times \mathrm{S}^d_{[0,r_1]}$, and then use pushforward $(\mathcal{J}^{-1})_*$.

$$\left\langle (\mathcal{J}^{-1})_*\nu, \Phi \right\rangle = \left\langle \nu, \Phi\big(\cdot, \mathcal{J}^{-1}(\cdot)\big) \right\rangle ,$$

where ν is a distribution on $\Omega \times \mathrm{S}^d_{[0,r_1]}$.

Richard Melrose presents three different definitions of distributions on even more general manifolds (with corners).
For a smooth compact manifold with corners X let us denote by ΩX a C^∞ line bundle over X consisting of densities (1-densities). The spaces

$$\left(\mathrm{C}_c^\infty(\mathsf{Int}\, X;\Omega X)\right)', \quad \left(\mathrm{C}^\infty(X;\Omega X)\right)', \quad \text{and} \quad \left(\mathrm{C}_0^\infty(X;\Omega X)\right)'$$

are called *distributions in the interior*, *supported distributions* and *extendible distributions*, respectively. Here $\mathrm{C}_0^\infty(X;\Omega X)$ denotes smooth functions which vanish, with all derivatives, at the boundary of X.

Since $\mathrm{S}^d_{\langle 0,1]}$ is open in $\mathbf{R}^d$, on $\mathrm{S}^d_{[0,r_1]}$ we have a canonical way how to integrate differential d-forms, thus in our situation densities can be omitted.
Furthermore, we want to take $\mathrm{C}^\infty(\mathrm{S}^d_{[0,r_1]})$ (i.e. $\mathrm{C}^\infty(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$) for the space of test functions in the dual space, thus we shall always use supported distributions on $\mathrm{S}^d_{[0,r_1]}$ (and hence on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$).
Since $\mathrm{C}_c^\infty(\mathrm{S}^d_{[0,r_1]}) = \mathrm{C}^\infty(\mathrm{S}^d_{[0,r_1]})$, one can see supported distributions as a natural extension of (standard) distributions to compact sets. Thus, we shall keep the same notation: $\mathcal{D}'(\mathrm{S}^d_{[0,r_1]}) = (\mathrm{C}^\infty(\mathrm{S}^d_{[0,r_1]}))'$. Moreover, it is straightforward to see that our notion of anisotropic distributions can be generalised to supported distributions.

Therefore, we shall use the following notation:
$\mathcal{D}_{l,m}(\Omega\times\mathrm{K}_{0,\infty}(\mathbf{R}^d)) := (\mathrm{C}_c^{l,m}(\Omega\times\mathrm{K}_{0,\infty}(\mathbf{R}^d)))'$.

The kernel theorem

Lemma. Let X and Y be smooth manifolds without boundary, of dimension d and r, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$, and B a continuous bilinear form on $\mathrm{C}^l_c(X) \times \mathrm{C}^m_c(Y)$.
Then there exists a unique distribution of anisotropic order $\nu \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$ such that

$$\big(\forall f \in \mathrm{C}^l_c(X)\big)\big(\forall g \in \mathrm{C}^m_c(Y)\big) \quad B(f,g) = \langle \nu, f \otimes g \rangle \ .$$

We extend it to $Z \subseteq \mathbf{R}^d_+ := \{\mathbf{x} = (\mathbf{x}', x_d) \in \mathbf{R}^d : x_d \geq 0\}$.
The restrictions of smooth functions from $\mathbf{R}^d$ to $\mathbf{R}^d_+$ preserve smoothness.
The converse is also fulfilled, but we cannot use a simple extension by reflection, which suffices for continuous functions but we use *the Seeley extension* which is just a linear version of a more general result given by Whitney.

Lemma. For open $\tilde{\Omega} \subseteq \mathbf{R}^d$ let $\Omega := \tilde{\Omega} \cap \mathbf{R}^d_+$. Then there exists continuous linear mapping $E : \mathrm{C}^\infty(\Omega) \to \mathrm{C}^\infty(\tilde{\Omega})$ such that for any $\psi \in \mathrm{C}^\infty(\Omega)$ we have $E(\psi)_{|\Omega} = \psi$.
Of course, if ψ has a compact support (in Ω), then we can choose $E(\psi)$ such that it has also compact support (in $\tilde{\Omega}$).

The kernel theorem (cont.)

Now we can repeat standard arguments regarding constructions on manifolds with boundary [N.A., M. Erceg, M. Lazar], obtaining the following result.

Theorem. Let $\Omega \subseteq \mathbf{R}^d$ be open, $l, m \in \mathbf{N} \cup \{\infty\}$, and B be a continuous bilinear form on $\mathrm{C}^l_c(\Omega) \times \mathrm{C}^m(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$. Then there exists a unique supported distribution of anisotropic order $\nu \in \mathcal{D}'_{l,d(m+2)}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that

$$\big(\forall f \in \mathrm{C}^l_c(\Omega)\big)\big(\forall g \in \mathrm{C}^{d(m+2)}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))\big) \quad B(f,g) = \langle \nu, f \otimes g \rangle \ .$$

Alternatively, we could embed $\mathrm{S}^d_{[0,r_1]}$ into torus [R. Melrose], and then apply directly the first representation.

One-scale H-distributions

Teorem. If $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$ and (v_n) is bounded in $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$, for some $p \in \langle 1, \infty\rangle$ and $q \geqslant p'$, and $\omega_n \to 0^+$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a complex valued (supported) distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'_{0,K}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$, where $K := d(\kappa+2)$, such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}^K(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we have

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\overline{\varphi_2 v_{n'}}\, d\mathbf{x} = \lim_{n'} \Big\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\Big\rangle$$

$$= \Big\langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi\Big\rangle ,$$

where $\psi_n := \psi(\omega_n \cdot)$. The distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ we call one-scale H-distribution (with characteristic length $(\omega_{n'})$) associated to (sub)sequences $(u_{n'})$ and $(v_{n'})$.

The existence of one-scale H-distributions: proof

For $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ and $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ such that $\operatorname{supp}\varphi_1, \operatorname{supp}\varphi_2 \subseteq K_m$, we have

$$|\Big\langle \varphi_2 v_n, \mathcal{A}_{\psi_n}(\varphi_1 u_n)\Big\rangle| \leqslant C_{m,d}\|\varphi_1\|_{\mathrm{L}^\infty(K_m)}\|\varphi_2\|_{\mathrm{L}^\infty(K_m)}\|\psi\|_{\mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}\,,$$

where K_m are compacts such that $K_m \subseteq \operatorname{Int} K_{m+1}$ and $\bigcup_m K_m = \Omega$.

By the Cantor diagonal procedure (in a separable space) we get a *trilinear* form

$$L(\varphi_1, \varphi_2, \psi) = \lim_{n'}\Big\langle \overline{\varphi_2 v_{n'}}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\Big\rangle\,,$$

which depends only on the product $\varphi_1\bar\varphi_2$, by the Commutation lemma.

Indeed, take $\zeta_i \equiv 1$ on $\operatorname{supp}\varphi_i$

$$\begin{aligned}
\lim_{n'}\Big\langle \overline{\varphi_2 v_{n'}}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})\Big\rangle &= \lim_{n'}\Big\langle \overline{\varphi_2 v_{n'}}, \varphi_1\mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\Big\rangle \\
&= \lim_{n'}\Big\langle \overline{\bar\varphi_1\varphi_2 v_{n'}}, \mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\Big\rangle \\
&= \lim_{n'}\Big\langle \overline{\zeta_1\zeta_2 v_{n'}}, \varphi_1\bar\varphi_2\mathcal{A}_{\psi_{n'}}(\zeta_1 u_n)\Big\rangle \\
&= \lim_{n'}\Big\langle \overline{\zeta_1\zeta_2 v_{n'}}, \mathcal{A}_{\psi_{n'}}(\varphi_1\bar\varphi_2 u_n)\Big\rangle\,,
\end{aligned}$$

For $\varphi \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}^K(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ we define

$$B(\varphi, \psi) := L(\varphi, \zeta, \psi)\,.$$

The existence of one-scale H-distributions: proof (cont.)

B is a continuous bilinear form on $\mathrm{C}_c(\Omega) \times \mathrm{C}^K(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, satisfying $B(\varphi_1\bar{\varphi}_2, \psi) = L(\varphi_1, \varphi_2, \psi)$.

Now we can apply the Kernel theorem, which gives us that there exists $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'_{0,K}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that

$$\begin{aligned}
\left\langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \right\rangle &= B(\varphi_1\bar{\varphi}_2, \psi) \\
&= L(\varphi_1\bar{\varphi}_2, \zeta_1\zeta_2, \psi) \\
&= L(\varphi_1, \varphi_2, \psi) = \lim_{n'} \left\langle \varphi_2 v_{n'}, \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'}) \right\rangle ,
\end{aligned}$$

as required.

Example 4: oscillations - two characteristic length

$0 < \alpha < \beta$, $\mathsf{k}, \mathsf{s} \in \mathbf{Z}^d \setminus \{0\}$,

$$u_n(\mathbf{x}) := e^{2\pi i (n^\alpha \mathsf{s} + n^\beta \mathsf{k}) \cdot \mathbf{x}} \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0\,,\ n \to \infty$$

$$\mu_H = \lambda \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$$

$$\mu_{\mathrm{K}_{0,\infty}}^{(\omega_n)} = \lambda \boxtimes \begin{cases} \delta_{0^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\beta \omega_n = 0 \\ \delta_{c\mathsf{k}} & , \quad \lim_n n^\beta \omega_n = c \in \langle 0, \infty \rangle \\ \delta_{\infty^{\frac{\mathsf{k}}{|\mathsf{k}|}}} & , \quad \lim_n n^\beta \omega_n = \infty \end{cases}$$

Lower order term n^α and corresponding direction of oscillations s we cannot resemble in any case.
Therefore, we need some new methods and/or tools.

- L. Tartar: *Multi-scale H-measures,* Discrete and Continuous Dynamical Systems - Series S **8** (2015) 77–90.

Still no satisfactory results.

Thank you for your attention!