H-distributions and variants

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MiTPDE

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Existence of H-measures

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2{}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^r)$, *then there exists its subsequence and a complex matrix Radon measure distribution of order zero* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi})\,.$$

∎

There are some other variants: (ultra)parabolic, fractional, one-scale, ...
Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^d)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^d)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x}) u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier $\mathcal{A}_a$, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{\mathcal{A}_a u} = a\hat{u}$.
The norm is again equal to $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.
Delicate part: a is given only on S^1.
We extend it by the projection p: if α is a function defined on a compact surface, we take $a := \alpha \circ p$, i.e.

$$a(\boldsymbol{\xi}) := \alpha\Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)$$

The precise scaling is contained in the projections, not the surface.

Good bounds in the L^p case: the Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi \mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)\ , \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi \mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

can be extended to a continuous mapping $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$.

Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2} \leqslant |\boldsymbol{\xi}| \leqslant r} |\partial^{\boldsymbol{\alpha}} \psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|}\ ,$$

then for any $p \in \langle 1, \infty \rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p \to \mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty)\ .$$

∎

For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

Existence of H-distributions

Theorem. *(N.A. & D. Mitrović, 2011) If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$ *and* $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^q(\mathbf{R}^d)$ *for some* $q \geqslant \max\{p', 2\}$*, then there exist subsequences* $(u_{n'})$*,* $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$*, such that for every* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ *we have:*

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} &= \lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} \\ &= \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle \,,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$ *is the Fourier multiplier operator with symbol* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$*.* ∎

We call the functional μ the *H-distribution* corresponding to (a subsequence of) (u_n) and (v_n).

For applications, of interest is to extend the result to vector-valued functions. For $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

In contrast to H-measures, we cannot consider H-distributions corresponding to the same sequence, but only to a pair of sequences, and H-distribution would correspond to non-diagonal blocks for H-measures.

First commutation lemma

$\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ satisfies the conditions of the Hörmander-Mihlin theorem.
Therefore, $\mathcal{A}_\psi$ and B are bounded operators on $\mathrm{L}^r(\mathbf{R}^d)$, for any $r \in \langle 1, \infty \rangle$.
We are interested in the properties of their commutator, $C = \mathcal{A}_\psi B - B\mathcal{A}_\psi$.
If $p < r$, we can apply the classical interpolation inequality:

$$\|Cv_n\|_p \leqslant \|Cv_n\|_2^\alpha \|Cv_n\|_r^{1-\alpha} ,$$

for $\alpha \in \langle 0, 1 \rangle$ such that $1/p = \alpha/2 + (1-\alpha)/r$.
As C is compact on $\mathrm{L}^2(\mathbf{R}^d)$ by Tartar's First commutation lemma, while it is bounded on $\mathrm{L}^r(\mathbf{R}^d)$, we get the claim.

For the most interesting case, where $p = r$, we need a better result: the Krasnosel'skij theorem (in fact, its extension to unbounded domains [N.A., M. Mišur, D. Mitrović (2018)]).

Lemma. *Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty \rangle \setminus \{2\}$. Then A is also compact on any $\mathrm{L}^p(\mathbf{R}^d)$, where $1/p = \theta/2 + (1-\theta)/r$, for a $\theta \in \langle 0, 1 \rangle$.* ∎

Therefore, the commutator C is compact on all $\mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1, \infty \rangle$.

Lemma on bilinear forms

Lemma. *Let E, F be separable Banach spaces, (b_n) an equibounded sequence of bilinear forms on $E \times F$ (i.e. $|b_n(\varphi, \psi)| \le C\|\varphi\|_E\|\psi\|_F$). Then there exists a subsequence (b_{n_k}) and a bilinear form b (with the same bound C) such that*

$$(\forall \varphi \in E)(\forall \psi \in F) \qquad \lim_k b_{n_k}(\varphi, \psi) = b(\varphi, \psi) \; .$$

∎

$\mathrm{C}_{K_l}(\mathbf{R}^d) := \{\varphi \in \mathrm{C}(\mathbf{R}^d) : \operatorname{supp} \varphi \subseteq K_l\}$ and

$$\mathrm{C}_c(\mathbf{R}^d) = \bigcup_{l \in \mathbf{N}} \mathrm{C}_{K_l}(\mathbf{R}^d) \; ,$$

so we have $B^l \in \mathcal{L}(\mathrm{C}_{K_l}(\mathbf{R}^d); (\mathrm{C}^\kappa(\mathrm{S}^{d-1}))')$, and we can keep the convergence on $\mathrm{C}_{K_{l-1}}(\mathbf{R}^d)$, in such a way obtaining that B^l is an extension of B^{l-1}.

Thus define B on $\mathrm{C}_c(\mathbf{R}^d)$:
for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$ we take $l \in \mathbf{N}$ such that $\operatorname{supp} \varphi \subseteq K_l$, and set $B\varphi := B^l\varphi$.

The definition is good, and we have a bounded operator in uniform norm:

$$\|B\varphi\|_{(\mathrm{C}^\kappa(\mathrm{S}^{d-1}))'} \leqslant \tilde{C}\|\varphi\|_{\mathrm{C}_0(\mathbf{R}^d)} \; .$$

It can be extended to the completion, the Banach space $\mathrm{C}_0(\mathbf{R}^d)$.

Complete the proof . . .

Now we can define $\mu(\varphi, \psi) := \langle B\varphi, \psi\rangle$, which satisfies the Theorem.

Indeed, restrict B to $\mathrm{C}_c^\infty(\mathbf{R}^d)$; the restriction $\tilde{B}$ remains continuous. $(\mathrm{C}^\kappa(\mathrm{S}^{d-1}))'$ is a subspace of $\mathcal{D}'(\mathrm{S}^{d-1})$, and we have a continuous operator from $\mathrm{C}_c^\infty(\mathbf{R}^d)$ to $\mathcal{D}'(\mathrm{S}^{d-1})$, which by the Schwartz kernel theorem can be identified to a distribution from $\mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$.

However, the bounds we had indicate that we should have a better object than just a distribution, say of order no more than $\kappa = [d/2] + 1$.

(Un)fortunately, the situation is much more complicated. Just to mention that the specific examples of H-distributions that we had (up to recently) were all of order 0 in both variables.

It remains to:

◦ Make precise the anisotropic order of a distribution.

◦ Get a more precise form of the Schwartz kernel theorem.

Functions of anisotropic smoothness

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$ (or C^∞ manifolds), $\Omega \subseteq X \times Y$.

By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, if $|\boldsymbol{\alpha}| \leqslant l$ and $|\boldsymbol{\beta}| \leqslant m$,

$$\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega) \ .$$

$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p_{K_n}^{l,m}(f) := \max_{|\boldsymbol{\alpha}|\leqslant l, |\boldsymbol{\beta}|\leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} \ ,$$

where $K_n \subseteq \Omega$ are compacts, such that $\Omega = \cup_{n\in\mathbf{N}} K_n$ and $K_n \subseteq \mathrm{Int} K_{n+1}$.

For a compact set $K \subseteq \Omega$ we define a subspace of $\mathrm{C}^{l,m}(\Omega)$

$$\mathrm{C}_K^{l,m}(\Omega) := \left\{ f \in \mathrm{C}^{l,m}(\Omega) : \ \mathrm{supp}\, f \subseteq K \right\} .$$

This subspace inherits the topology from $\mathrm{C}^{l,m}(\Omega)$, which is, when considered only on the subspace, a norm topology determined by

$$\|f\|_{l,m,K} := p_K^{l,m}(f) \ ,$$

and $\mathrm{C}_K^{l,m}(\Omega)$ is a Banach space (it can be identified with a proper subspace of $\mathrm{C}^{l,m}(K)$). However, if $m = \infty$ (or $l = \infty$), then we shall not get a Banach space, but a Fréchet space. As in the isotropic case, an increasing sequence of seminorms that makes $\mathrm{C}_{K_n}^{l,\infty}(\Omega)$ a Fréchet space is given by $(p_{K_n}^{l,k}), k \in \mathbf{N}_0$.

Functions of anisotropic smoothness (cont.)

We can also consider the space

$$\mathrm{C}_c^{l,m}(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}_{K_n}^{l,m}(\Omega)\ ,$$

of all functions with compact support in $\mathrm{C}^{l,m}(\Omega)$, and equip it by a stronger topology than the one induced from $\mathrm{C}^{l,m}(\Omega)$: by the topology of *strict inductive limit*.

More precisely, it can easily be checked that

$$\mathrm{C}_{K_n}^{l,m}(\Omega) \hookrightarrow \mathrm{C}_{K_{n+1}}^{l,m}(\Omega)\ ,$$

the inclusion being continuous. Also, the topology induced on $\mathrm{C}_{K_n}^{l,m}(\Omega)$ by that of $\mathrm{C}_{K_{n+1}}^{l,m}(\Omega)$ coincides with the original one, and $\mathrm{C}_{K_n}^{l,m}(\Omega)$ (as a Banach space in that topology) is a closed subspace of $\mathrm{C}_{K_{n+1}}^{l,m}(\Omega)$. Then we have that the strict inductive limit topology on $\mathrm{C}_c^{l,m}(\Omega)$ induces on each $\mathrm{C}_{K_n}^{l,m}(\Omega)$ the original topology, while a subset of $\mathrm{C}_c^{l,m}(\Omega)$ is bounded if and only if it is contained in one $\mathrm{C}_{K_n}^{l,m}(\Omega)$, and bounded there. $\mathrm{C}_c^{l,m}(\Omega)$ is a barelled space.

Of course, $\mathrm{C}_c^{\infty}(\Omega) \hookrightarrow \mathrm{C}_c^{l,m}(\Omega)$ is a continuous and dense imbedding.

Distributions of anisotropic order

Definition. A *distribution of order l in $\mathbf{x}$ and order m in $\mathbf{y}$* is any linear functional on $\mathrm{C}_c^{l,m}(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Clearly, $\mathrm{C}_c^{\infty}(\Omega) \hookrightarrow \mathrm{C}_c^{l,m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$, with continuous and dense imbeddings, thus $\mathrm{C}_c^{l,m}(\Omega)$ is a normal space of distributions, hence its dual $\mathcal{D}'_{l,m}(\Omega)$ forms a subspace of $\mathcal{D}'(\Omega)$. If we equip it with a strong topology, it is even continuously imbedded in $\mathcal{D}'(\Omega)$.

Lemma. *Let X and Y be C^{∞} manifolds. For a linear functional u on $\mathrm{C}_c^{l,m}(X \times Y)$, the following statements are equivalent*

a) $u \in \mathcal{D}'_{l,m}(X \times Y)$,

b) $(\forall K \in \mathcal{K}(X \times Y))(\exists C > 0)(\forall \Psi \in \mathrm{C}_K^{l,m}(X \times Y)) \quad |\langle u, \Psi \rangle| \leqslant C p_K^{l,m}(\Psi)$. ∎

Statement (b) of previous lemma implies:

$$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^{l}(X))(\forall \psi \in \mathrm{C}_L^{m}(Y))$$
$$|\langle u, \varphi \boxtimes \psi \rangle| \leqslant C p_K^{l}(\varphi) p_L^{m}(\psi) \, .$$

The reverse implication would have significantly greater practical use.

Schwartz kernel theorem

Theorem. *Let X and Y be two differentiable manifolds.*

a) *Let $K \in \mathcal{D}'_{l,m}(X \times Y)$. Then for each $\varphi \in \mathrm{C}^l_c(X)$ the linear form K_φ, defined by $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$, is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ with its inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.*

b) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator, in the pair of topologies as above. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathrm{C}^\infty_c(X)$ and $\psi \in \mathrm{C}^\infty_c(Y)$*

$$\langle K, \varphi_{\boxtimes} \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

Furthermore, $K \in \mathcal{D}'_{l,d(m+2)}(X \times Y)$. ∎

Remarks

Note that in part (b) we did not get $K \in \mathcal{D}'_{l,m}(X \times Y)$, as one would expect. The order with respect to $\mathbf{x}$ variable remained the same, but the order with respect to $\mathbf{y}$ increased from m to $d(m+2)$. Interchanging the roles of X and Y, the same proof gives $K \in \mathcal{D}'_{d(l+2),m}(X \times Y)$, where order with respect to $\mathbf{y}$ remained the same, but order with respect to the $\mathbf{x}$ variable increased from l to $d(l+2)$. Since uniqueness of $K \in \mathcal{D}'(X \times Y)$ has already been determined, we conclude that $K \in \mathcal{D}'_{l,d(m+2)}(X \times Y) \cap \mathcal{D}'_{d(l+2),m}(X \times Y)$. It might be interesting to see some additional properties of that intersection.

If one used a more constructive proof of the Schwartz kernel theorem, for example [Simanca, Theorem 1.3.4], one would end up increasing the order with respect to both variables $\mathbf{x}$ and $\mathbf{y}$. This occurs naturally, because one needs to secure the integrability of the function which is used to define the kernel function.

Consequence for H-distributions

By the previous theorem the H-distribution μ mentioned at the beginning belongs to the space $\mathcal{D}'_{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than $d(\kappa+2)$ in $\boldsymbol{\xi}$.

Indeed, we already have $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ and the following bound with $\varphi := \varphi_1 \overline{\varphi_2}$:

$$|\langle \mu, \varphi \boxtimes \psi \rangle| \leqslant C \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} \,,$$

where C does not depend on φ and ψ.

Now we just need to apply the Schwartz kernel theorem given above to conclude that μ is a continuous linear functional on $\mathrm{C}_c^{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$.

Example: H-distributions are not measures

In order to explicitly compute the H-distribution in some nontrivial case, it is advantageous to relate it to only one sequence.

Canonical choice of an $\mathrm{L}^{p'}$ sequence corresponding to an L^p, $p \in \langle 1, \infty\rangle$, sequence (u_n) is given by $v_n = \Phi_p(u_n)$, where Φ_p is an operator from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\Phi_p(u) = |u|^{p-2}u$.

Φ_p is a nonlinear Nemytskij operator, continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ and additionally we have the following bound

$$\|\Phi_p(u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} \leqslant \|u\|^{p/p'}_{\mathrm{L}^p(\mathbf{R}^d)}.$$

It maps bounded sets in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ topology to bounded sets in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ topology. Hence for an L^p bounded sequence (u_n), we get that $(\Phi_p(u_n))$ is weakly precompact in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

It is continuous from $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

Example: concentration

$u \in \mathrm{L}^p_c(\mathbf{R}^d)$, and define $u_n(\mathbf{x}) = n^{\frac{d}{p}} u(n(\mathbf{x}-\mathbf{z}))$ for some $\mathbf{z} \in \mathbf{R}^d$.

Simple change of variables: $\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)} = \|u\|_{\mathrm{L}^p(\mathbf{R}^d)}$ and $u_n \rightharpoonup 0$ in $\mathrm{L}^p(\mathbf{R}^d)$.

Indeed, the sequence is bounded, while for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$

$$\begin{aligned}
\int_{\mathbf{R}^d} u_n(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} &= \int_{\mathbf{R}^d} n^{d/p} u(n(\mathbf{x}-\mathbf{z}))\varphi(\mathbf{x})d\mathbf{x} \\
&= \int_{\mathbf{R}^d} n^{d/p-d} u(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\
&= \frac{1}{n^{d/p'}} \int_{\mathbf{R}^d} u(\mathbf{y})\chi_{\mathrm{supp}\, u}(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\
&\leqslant \left(\frac{\mathrm{vol}(\mathrm{supp}\, u)}{n^d}\right)^{1/p'} \|u\|_{\mathrm{L}^p(\mathbf{R}^d)} \max_{\mathbf{R}^d} |\varphi|.
\end{aligned}$$

Passing to the limit, we get our claim.

The H-distribution corresponding to sequences (u_n) and $(\Phi_p(u_n))$ is given by $\delta_{\mathbf{z}} \boxtimes \nu$, where ν is a continuous functional on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$\langle \nu, \psi \rangle = \int_{\mathbf{R}^d} u(\mathbf{x}) \overline{\mathcal{A}_{\bar{\psi}}(|u|^{p-2}u)(\mathbf{x})} d\mathbf{x}.$$

This distribution might be a Radon measure, or not.

The operators $\zeta_{p,n}$

For $p \in \langle 1, \infty \rangle$, $\mathbf{z} \in \mathbf{R}^d$, and $n \in \mathbf{N}$, define a linear operator $\zeta_{p,n}$ on $\mathrm{L}^p(\mathbf{R}^d)$

$$\zeta_{p,n} u(\mathbf{x}) = n^{\frac{d}{p}} u(n(\mathbf{x} - \mathbf{z}))\,.$$

∘ It is a linear isometry on $\mathrm{L}^p(\mathbf{R}^d)$, i.e.

$$\|\zeta_{p,n} u\|_{\mathrm{L}^p(\mathbf{R}^d)} = \|u\|_{\mathrm{L}^p(\mathbf{R}^d)}\,.$$

∘ For any $u \in \mathrm{L}^p(\mathbf{R}^d)$ the sequence $(\zeta_{p,n} u)$ weakly converges to 0 in $\mathrm{L}^p(\mathbf{R}^d)$. Indeed, for u with a compact support, since $(\zeta_{p,n} u)$ is bounded, it is sufficient to take a continuous test function φ with compact support

$$\begin{aligned}
\int_{\mathbf{R}^d} \zeta_{p,n} u(\mathbf{x}) \varphi(\mathbf{x})\, d\mathbf{x} &= \int_{\mathbf{R}^d} n^{d/p} u(n(\mathbf{x} - \mathbf{z})) \varphi(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\mathbf{R}^d} n^{d/p - d} u(\mathbf{y}) \varphi(\mathbf{y}/n + \mathbf{z})\, d\mathbf{y} \\
&= \frac{1}{n^{d/p'}} \int_{\operatorname{supp} u} u(\mathbf{y}) \varphi(\mathbf{y}/n + \mathbf{z})\, d\mathbf{y} \\
&\leqslant \left(\frac{\mathrm{vol}(\operatorname{supp} u)}{n^d} \right)^{1/p'} \|u\|_{\mathrm{L}^p(\mathbf{R}^d)} \max_{\mathbf{R}^d} |\varphi|\,,
\end{aligned}$$

where we have used the change of variables $\mathbf{y} = n(\mathbf{x} - \mathbf{z})$ in the second equality and the Hölder inequality in the last step. Passing to the limit, we get our claim.

Φ_p and $\zeta_{p,n}$ commute

For arbitrary $u \in \mathrm{L}^p(\mathbf{R}^d)$ and $v \in \mathrm{L}^{p'}(\mathbf{R}^d)$, $1/p + 1/p' = 1$, we will show that the H-distribution corresponding to sequences $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$ is given by $\delta_{\mathbf{z}} \boxtimes \nu$, where ν is a functional on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$\langle \nu, \psi \rangle = \int_{\mathbf{R}^d} u(\mathbf{x}) \overline{\mathcal{A}_{\bar{\psi}} v(\mathbf{x})}\, d\mathbf{x}\,.$$

Φ_p and $\zeta_{p,n}$ commute in the following sense: for $u \in \mathrm{L}^p(\mathbf{R}^d)$

$$\begin{aligned}\Phi_p(\zeta_{p,n}u)(\mathbf{x}) &= |n^{\frac{d}{p}} u(n(\mathbf{x}-\mathbf{z}))|^{p-2} n^{\frac{d}{p}} u(n(\mathbf{x}-\mathbf{z})) \\ &= n^{\frac{d(p-1)}{p}} |u(n(\mathbf{x}-\mathbf{z}))|^{p-2} u(n(\mathbf{x}-\mathbf{z})) = \zeta_{p',n}\Phi_p(u)(\mathbf{x})\,,\end{aligned}$$

by taking $v = \Phi_p(u)$ we reveal the canonical choice of the $\mathrm{L}^{p'}$ sequence corresponding to $(\zeta_{p,n}u)$, i.e. $\zeta_{p',n}v = \Phi_p(\zeta_{p,n}u)$.

Before we proceed, we need two lemmata:

Lemma. *Let $p \in \langle 1, \infty\rangle$ and $\mathbf{z} \in \mathbf{R}^d$. For any $u \in \mathrm{L}^p(\mathbf{R}^d)$ and $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$ it holds*

$$\varphi\zeta_{p,n}u - \varphi(\mathbf{z})\zeta_{p,n}u \longrightarrow 0 \textit{ in } \mathrm{L}^p(\mathbf{R}^d)\,.$$

∎

. . . and the second lemma

Lemma. *For any $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, $p \in \langle 1, \infty\rangle$, $\mathbf{z} \in \mathbf{R}^d$, and $n \in \mathbf{N}$, the operators $\mathcal{A}_\psi$ and $\zeta_{p,n}$ commute on $\mathrm{L}^p(\mathbf{R}^d)$.* ∎

For $v \in \mathcal{S}(\mathbf{R}^d)$, we have

$$\begin{aligned}
\mathcal{A}_\psi(\zeta_{p,n}v)(\mathbf{x}) &= n^{\frac{d}{p}}\bar{\mathcal{F}}\Big(\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\int_{\mathbf{R}^d} e^{-2\pi i\mathbf{y}\cdot\boldsymbol{\xi}}v(n(\mathbf{y}-\mathbf{z}))\,d\mathbf{y}\Big)(\mathbf{x}) \\
&= n^{\frac{d}{p}}\bar{\mathcal{F}}\Big(n^{-d}e^{-2\pi i\mathbf{z}\cdot\boldsymbol{\xi}}\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\int_{\mathbf{R}^d} e^{-2\pi i\frac{\mathbf{w}}{n}\cdot\boldsymbol{\xi}}v(\mathbf{w})\,d\mathbf{w}\Big)(\mathbf{x}) \\
&= n^{\frac{d}{p}}n^{-d}\bar{\mathcal{F}}\Big(e^{-2\pi i\mathbf{z}\cdot\boldsymbol{\xi}}\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\hat{v}(\boldsymbol{\xi}/n)\Big)(\mathbf{x}) \\
&= n^{\frac{d}{p}}n^{-d}\int_{\mathbf{R}^d} e^{2\pi i(\mathbf{x}-\mathbf{z})\cdot\boldsymbol{\xi}}\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\hat{v}(\boldsymbol{\xi}/n)\,d\boldsymbol{\xi} \\
&= n^{\frac{d}{p}}\int_{\mathbf{R}^d} e^{2\pi i\boldsymbol{\eta}\cdot(n(\mathbf{x}-\mathbf{z}))}\psi(\boldsymbol{\eta}/|\boldsymbol{\eta}|)\hat{v}(\boldsymbol{\eta})\,d\boldsymbol{\eta} = \zeta_{p,n}(\mathcal{A}_\psi v)(\mathbf{x})\,.
\end{aligned}$$

Since $\mathcal{S}(\mathbf{R}^d)$ is dense in $\mathrm{L}^p(\mathbf{R}^d)$, while $\mathcal{A}_\psi$ and $\zeta_{n,p}$ are continuous on $\mathrm{L}^p(\mathbf{R}^d)$, we get the claim.

H-distribution corresponding to sequences $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$

Let us show that $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$ form a pure pair (for arbitrary u and v). Taking $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d)$, we get

$$
\begin{aligned}
\lim_n \int_{\mathbf{R}^d} \varphi_1(\mathbf{x})(\zeta_{p,n}u)(\mathbf{x})\overline{\mathcal{A}_{\bar\psi}(\varphi_2\zeta_{p',n}v)(\mathbf{x})}d\mathbf{x} &= \varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z}) \lim_n \int_{\mathbf{R}^d} (\zeta_{p,n}u)(\mathbf{x})\overline{\mathcal{A}_{\bar\psi}(\zeta_{p',n}v)(\mathbf{x})} \\
&= \varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z}) \lim_n \int_{\mathbf{R}^d} (\zeta_{p,n}u)(\mathbf{x})\overline{\zeta_{p',n}\mathcal{A}_{\bar\psi}(v)(\mathbf{x})}\, d\mathbf{x} \\
&= \varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z}) \lim_n \int_{\mathbf{R}^d} n^d u(n(\mathbf{x}-\mathbf{z}))\overline{\mathcal{A}_{\bar\psi}(v)(n(\mathbf{x}-\mathbf{z}))}\, d\mathbf{x} \\
&= \varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z}) \lim_n \int_{\mathbf{R}^d} u(\mathbf{y})\overline{\mathcal{A}_{\bar\psi}(v)(\mathbf{y})}\, d\mathbf{y} \\
&= \left\langle u, \mathcal{A}_{\overline{\varphi_1(\mathbf{z})\bar\varphi_2(\mathbf{z})\psi}}(v) \right\rangle .
\end{aligned}
$$

The last expression can be extended by density to the whole $\mathrm{C}_c^{0,\kappa}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, thus we finally get that $(\zeta_{p,n}u)$ and $(\zeta_{p',n}v)$ form a pure pair, and the H-distribution is given by

$$
\langle \mu, \Psi \rangle = \left\langle u, \mathcal{A}_{\bar\Psi(\mathbf{z},\cdot)}(v) \right\rangle , \quad \Psi \in \mathrm{C}_c^{0,\kappa}(\mathbf{R}^d \times \mathrm{S}^{d-1}) .
$$

Its properties

In the example we got better (lower) order than those provided by the Theorem. In fact, the order $(0, \kappa)$, which is achieved in this example, is the best we can hope for H-distributions (exactly the bounds we have). The question of optimal κ, which is dictated by the Hörmander-Mihlin theorem, remains open.

Concering the value of κ, the case $\kappa = 0$ is particularly desirable since in that case we would have that H-distributions are Radon measures.

However, we shall show that *there are $u \in \mathrm{L}^p(\mathbf{R}^d)$ and $v \in \mathrm{L}^{p'}(\mathbf{R}^d)$ such that the H-distribution above is not a Radon measure*.

Indeed, take $\psi \in \mathrm{C}^\infty(\mathrm{S}^{d-1})$ such that $\|\psi\|_{\mathrm{L}^\infty(\mathrm{S}^{d-1})} = 1$. Then for any $n \in \mathbf{N}$ we have $\psi^n \in \mathrm{C}^\infty(\mathrm{S}^{d-1})$ and $\|\psi^n\|_{\mathrm{L}^\infty(\mathrm{S}^{d-1})} = 1$. By the Banach-Steinhaus theorem the uniform boundedness in n of $(\mathcal{A}_{\bar\psi^n})$ in $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d); \mathrm{L}^p(\mathbf{R}^d))$ (here we assume that $\bar\psi^n$ is extended to $\mathbf{R}^d \setminus \{0\}$ along rays through the origin, as usual) is equivalent to the propery that for any $u \in \mathrm{L}^p(\mathbf{R}^d)$ and $v \in \mathrm{L}^{p'}(\mathbf{R}^d)$ the sequence $\Big(|\langle u, \mathcal{A}_{\bar\psi^n} v\rangle|\Big)$ is bounded.

The first implication is trivial, while to prove the latter we first apply the uniform boundedness principle to $v \mapsto \langle u, \mathcal{A}_{\bar\psi^n} v\rangle$ (for an arbitrary u), and then to $\mathcal{A}_{\bar\psi^n}$.

Ahlfros-Beurling operator

Thus, it is sufficient to find $\psi \in \mathrm{C}^\infty(\mathrm{S}^{d-1})$, $\|\psi\|_{\mathrm{L}^\infty(\mathrm{S}^{d-1})} = 1$, such that $(\mathcal{A}_{\bar{\psi}^n})$ is not uniformly bounded in $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d); \mathrm{L}^p(\mathbf{R}^d))$, as then by the above equivalence the mapping $\psi \mapsto \langle u, \mathcal{A}_{\bar{\psi}} v\rangle$ cannot be continuous on $\mathrm{C}(\mathrm{S}^{d-1})$, implying that the H-distribution above is not a Radon measure.

For an example of such ψ in two space dimensions ($d = 2$) one can consider the symbol of the Ahlfros-Beurling operator (Dragičević, 2011) which is given by $\psi(\xi_1, \xi_2) = \xi_1 + i\xi_2$ since it is know that $\|\mathcal{A}_{\bar{\psi}^n}\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d); \mathrm{L}^p(\mathbf{R}^d))}$ goes to infinity as n tends to infinity (loc.cit.).
For a counterexample in higher dimensions, one could apply the method of dilatations (Oşekowski, 2012).

To conclude, with the argument above we have proved that there exist H-distributions which are not Radon measures. Therefore, our kernel theorem is really meaningful when applied on H-distributions.

Nevertheless, one could still think of whether the order can be improved when for the $\mathrm{L}^{p'}$ sequence (v_n) one takes precisely the canonical choice $(\Phi_p(u_n))$. Although the above counterexample does not say anything for this specific case, webelieve that even for such (v_n)-s in general H-distributions are not Radon measures.

Thank you for your attention!