Microlocal defect functionals: H-distributions

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Existence of H-measures

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2{}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^r)$, *then there exists its subsequence and a complex matrix Radon measure distribution of order zero* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi})\,.$$

∎

There are some other variants: (ultra)parabolic, fractional, one-scale, ...
Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^d)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^d)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x}) u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier $\mathcal{A}_a$, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{\mathcal{A}_a u} = a\hat{u}$.
The norm is again equal to $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.
Delicate part: a is given only on S^1.
We extend it by the projection p: if α is a function defined on a compact surface, we take $a := \alpha \circ p$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\Big)$$

The precise scaling is contained in the projections, not the surface.

First commutation lemma

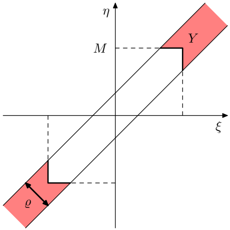
Lemma. (general form of the first commutation lemma — Luc Tartar)
If $b \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ *satisfy the condition*

$$(\forall\, \rho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})| \leqslant \varepsilon \ \ (\text{a.e. } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \rho))\ ,$$

then $C := [\mathcal{A}_a, M_b]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$*.* ∎

For given $M, \rho \in \mathbf{R}^+$ denote the set

$$Y = Y(M, \rho) = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geqslant M \ \&\ |\boldsymbol{\xi} - \boldsymbol{\eta}| \leqslant \rho\}\ .$$



[Some improvements in N.A., M. Mišur, D. Mitrović (MJOM, 2018); older results by H. O. Cordes (JFA, 1975)]

The importance of First commutation lemma

If we take $\mathsf{u}_n = (u_n, v_n)$, and consider $\mu = \mu_{12}$, we have

$$\begin{aligned}
\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} \psi \, d\boldsymbol{\xi} &= \lim_{n'} \langle \mathcal{A}_\psi(\varphi_1 u_{n'}) | \varphi_2 v_{n'} \rangle \\
&= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} \, d\mathbf{x} \\
&= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} \, d\mathbf{x} = \langle \mu, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle \,.
\end{aligned}$$

Thus the limit is a bilinear functional in $\varphi_1 \bar{\varphi}_2$ and ψ, and we have the bound:

$$\Big| \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} d\mathbf{x} \Big| \leqslant C \|\psi\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|\varphi_1 \overline{\varphi_2}\|_{\mathrm{C}_0(\mathbf{R}^d)} \,.$$

This form makes sense even for $p < 2$ (for $p > 2$ we use the fact that $u_n \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$).

A class of symbols (L. Tartar)

Actually, we can consider more general operators than $\mathcal{A}_a$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|b_m\|_{\mathrm{C}_0(\mathbf{R}^d)} = k < \infty$.

To such a symbol s, a *standard operator* $S_s \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$ is assigned by

$$S_s = \sum_m \mathcal{A}_{\alpha_m} M_b \ ,$$

with $\|S_s\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$.

Clearly, S_s does not depend on the above decomposition, as

$$\widehat{S_s u}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} s(\mathbf{x}, \boldsymbol{\xi}/|\boldsymbol{\xi}|) u(\mathbf{x})\, d\mathbf{x} \ ,$$

for u in a dense set of $\mathrm{L}^2(\mathbf{R}^d)$ (e.g. $\mathcal{S}$).

A class of symbols (cont.)

Any operator $A \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, which differs from S_s only by a compact operator, is an *operator of symbol* s, like

$$L_s = \sum_m M_{b_m} \mathcal{A}_{\alpha_m} \ ,$$

where $\|L_s\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$. Neither L_s depends on the decomposition.

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^r)$, *then there exists its subsequence and an H-measure* $\boldsymbol{\mu}$, *which is a Hermitian non-negative* $r \times r$ *matrix of distributions of order zero on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and any operators* $L_{s_1}, L_{s_2} \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, *with symbols* s_1, s_2 *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} L_{s_1}(\varphi_1 u^j_{n'}) \overline{L_{s_2}(\varphi_2 u^k_{n'})} \, d\boldsymbol{\xi} = \langle \mu^{jk}, \varphi_1 s_1 \overline{\varphi_2 s_2} \rangle \ .$$

∎

P. Gérard used a different approach, by using classical symbols. However, it is important to have symbols of lower regularity, as they come in applications from coefficients in PDEs.

We can consider $\Omega \subseteq \mathbf{R}^d$ as a domain, or even a manifold (with a volume form).

Symmetric systems

$$\sum_k \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \;\; , \;\; \mathbf{A}^k \text{ Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0 \quad \text{(weakly)},$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

If supports of $\mathsf{u}^n, \mathsf{f}^n$ are contained inside Ω, we can extend them by zero to $\mathbf{R}^d$.

Theorem. (localisation property) *If* $\mathsf{u}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)^r$ *defines* $\boldsymbol{\mu}$*, and if* u^n *satisfies:*

$$\partial_k\left(\mathbf{A}^k \mathsf{u}^n\right) \to 0 \;\text{ in the space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \;,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\Omega \times S^{d-1}$ *it holds:*

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}^\top = \mathbf{0} \; .$$

∎

Thus, the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{P}$ is a singular matrix.)

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \le m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi} \,.$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish in infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).
On the other hand, $\mathrm{H}^s(\mathbf{R}^d) \subseteq X^m$, for $s > m + \frac{d}{2}$.
X^m is an algebra with respect to the multiplication of functions; it holds:

$$\|f * g\|_{\mathrm{L}^1} \le \|f\|_{\mathrm{L}^1} \|g\|_{\mathrm{L}^1}$$

$$\|\hat{f} \cdot \hat{g}\|_{X^0} \le \|\hat{f}\|_{X^0} \|\hat{g}\|_{X^0}$$

$X^m_{\mathrm{loc}}(\Omega)$: the space of all functions u such that $\varphi u \in X^m$, for $\varphi \in \mathrm{C}^\infty_c(\Omega)$.

Lemma. *Let $\mathcal{A}_\alpha$, M_b be standard operators, with symbols α, b, such that $\alpha \in \mathrm{C}^1(S^{d-1})$ and $b \in X^1$.*
Then $C := [\mathcal{A}_\alpha, M_b] \in \mathcal{L}\big(\mathrm{L}^2(\mathbf{R}^d), \mathrm{H}^1(\mathbf{R}^d)\big)$, and ∇C has a symbol $(\nabla_{\boldsymbol{\xi}} \alpha \cdot \nabla_{\mathbf{x}} b)\boldsymbol{\xi}$.
(we extend α to a homogeneous function on $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{\mathbf{0}\}$) ∎

▸▸ Skip 2nd comm lemma

A smaller class of symbols (L. Tartar)

Corollary. *Under the above assumptions,*

$$\mathcal{A}_\alpha M_b \partial_j u = M_b \partial_j (\mathcal{A}_\alpha u) + Lu, \quad u \in \mathrm{L}^2(\mathbf{R}^d) ,$$

where L has a symbol $\xi_j\{\alpha, b\}$. ∎

Actually, we can consider more general operators than $\mathcal{A}_\alpha$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}^1(\mathrm{S}^{d-1})} \|b_m\|_{X^1} < \infty$,
and *standard operators* $S_s = \sum_m \mathcal{A}_{\alpha_m} M_b$.

Lemma. *If S_1, S_2 are standard operators with symbols s_1, s_2 as above, then*

$$\frac{\partial}{\partial x^j}[S_1, S_2] \quad \textit{has symbol} \quad \xi_j\{s_1, s_2\} .$$

∎

Propagation property for symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \ , \qquad \mathbf{A}^k \text{ Hermitian}$$

Theorem. *Let* $\mathbf{A}^k \in \mathrm{C}_0^1(\Omega; \mathrm{M}_{r\times r})$*.*
If $(\mathsf{u}^n, \mathsf{f}^n)$ *satisfy the above for* $n \in \mathbf{N}$*, and* $\mathsf{u}^n, \mathsf{f}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\Omega)$*,*
then for any $\psi \in \mathrm{C}_0^1(\Omega \times S^{d-1})$*, the H-measure associated to sequence* $(\mathsf{u}^n, \mathsf{f}^n)$*:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix},$$

satisfies:

$$\left\langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k - 2\psi \mathbf{S} \right\rangle + \langle 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}, \psi \rangle = 0 \ ,$$

where $\mathbf{S} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^*)$*, while the Poisson bracket is:*
$\{\phi, Q\} = \nabla_{\boldsymbol{\xi}} \phi \cdot \nabla_{\mathbf{x}} Q - \nabla_{\mathbf{x}} \phi \cdot \nabla_{\boldsymbol{\xi}} Q.$ *[Recall:* $\mathbf{P} = \xi_k \mathbf{A}^k$ *]* ∎

$\boldsymbol{\mu}$ is associated to the pair of sequences $(\mathsf{u}^n, \mathsf{f}^n)$, the block $\boldsymbol{\mu}_{11}$ is determined by u^n, $\boldsymbol{\mu}_{22}$ with f^n, while the non-diagonal blocks correspond to the product of u^n and f^n.

The equation for H-measure

Corollary. *In the sense of distributions on* $\Omega \times S^{d-1}$ *the H-measure* $\boldsymbol{\mu}$ *satisfies:*

$$\partial^l \mathbf{P} \cdot \partial_l \boldsymbol{\mu}_{11} - \partial_t^l (\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) \xi^l \\ + (2\mathbf{S} - \partial_l \mathbf{A}^l) \cdot \boldsymbol{\mu}_{11} = 2\mathsf{Re}\,\mathsf{tr}\boldsymbol{\mu}_{12} \,,$$

where $\partial_t^l := \partial^l - \xi^l \xi_k \partial^k$ *is the tangential gradient on the unit sphere.* ∎

This allows us to investigate the behaviour of H-measures as solutions of initial-value problems, with appropriate initial conditions.
Besides the wave equations, there are applications to Maxwell's and Dirac's systems, even to the equations that change their type (like the Tricomi equation).

The wave equation

$$(\rho u')' - \mathrm{div}\,(\mathbf{A}\nabla u) = g\ .$$

It can be written as an equivalent symmetric system ($t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$):

$$\partial_0(\rho\partial_0 u) - \sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) = g\ .$$

By introducing: $v_j := \partial_j u$, for $j \in 0..d$, we obtain (Schwarz' symmetries!):

$$\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A} & \\ 0 & & & \end{bmatrix} \partial_0 \mathsf{v} + \sum_{i=1}^{d} \begin{bmatrix} 0 & -a^{i1} & \cdots & -a^{id} \\ -a^{i1} & & & \\ \vdots & & \mathbf{0} & \\ -a^{id} & & & \end{bmatrix} \partial_i \mathsf{v}$$

$$+ \begin{bmatrix} b^0 & b^1 & \cdots & b^d \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{v} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix}\ .$$

The symbol of differential operator is:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \xi_k \mathbf{A}^k(\mathbf{x}) = \begin{bmatrix} \xi_0 \rho & -(\mathbf{A}\boldsymbol{\xi}')^\top \\ -\mathbf{A}\boldsymbol{\xi}' & \xi_0 \mathbf{A} \end{bmatrix}\ .$$

Transport of H-measures associated to the wave equation

From the localisation property we can conclude that $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$. For the right hand side of the equation we have:

$$\langle \gamma, \varphi_1 \bar{\varphi}_2 \psi \rangle := \lim_n \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 v_{0,n}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 g_n}(\boldsymbol{\xi})} \psi \left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right) d\boldsymbol{\xi} \ .$$

Theorem. *On $\mathbf{R}^{d+1} \times S^d$ measure ν satisfies ($Q := \rho\xi_0^2 - \mathbf{A}\boldsymbol{\xi}' \cdot \boldsymbol{\xi}'$):*

$$\nabla_{\boldsymbol{\xi}} Q \cdot \nabla_{\mathbf{x}}(\xi_0 \nu) - Q\partial_0 \nu + \big(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \mathbf{I}\big) \nabla_{\mathbf{x}} Q \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)\big(\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}\big)(\xi_0 \nu) = 2\mathrm{Re}\, \gamma$$

∎

The equation can be written in a nicer form:

$$\{Q, \xi_0 \nu\} + (\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}) \big[\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)(\xi_0 \nu)\big] - Q\partial_0 \nu = 2\mathrm{Re}\, \gamma \ .$$

Recent Tartar's result (2017)

Theorem. *Let* $u^n \in \mathrm{C}^0(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{C}^1(0,T;\mathrm{L}^2(\Omega))$ *be a sequence of solutions of a "wave equation"*

$$(\rho(u^n)')' - \operatorname{div}(\mathbf{A}\nabla u^n) + S^k\partial_k u^n \longrightarrow 0 \qquad \textit{in } \mathrm{L}^2_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)\ ,$$

with $\rho, \mathbf{A}$ *in* $\mathrm{X}^1_{loc} \cap \mathrm{C}^2$, $\rho > 0$ *and* $\mathbf{A}$ *real positive definite (or replace it with its symmetric part, and subsume the lower order terms in the last term), and* S^k *be standard operators with symbols* s^k.
If $u^n \longrightarrow 0$ *in* $\mathrm{H}^1_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)$, *then* ∇u^n *corresponds to an H-measure* $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\pi$, *then*

$$Q\pi = 0\ ,$$

and

$$\Big\langle \pi, \{\Psi, Q\} + (\xi_k s^k + \xi_k \bar{s}^k)\Psi \Big\rangle = 0\ ,$$

for $\Psi \in \mathrm{C}^1_c(\langle 0,T\rangle \times \Omega \times \mathrm{S}^d)$. ∎

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w\ . \end{cases}$$

We have used D'Alembert's formula for solution, our approach and the approach of P. Gérad, obtaining the same result in this special case (which is treatable by both methods, and explicit calculations).
Physically important quantity is *energy density:*

$$d(t,x) := \frac{1}{2}(u_t^2 + u_x^2)\ ,$$

as well as the *energy* at time t: $$e(t) := \int_{\mathbf{R}} d(t,x)\, dx.$$

After simple calculations we get

$$4d(t,x) = \Big(v'(x+t) + w(x+t)\Big)^2 + \Big(v'(x-t) - w(x-t)\Big)^2\ .$$

Assume that the physical system is modelled by the above wave equation on the microscale. In order to pass to the macroscale, in the spirit of Tatar's programme, we have to pass to the weak limit.

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightharpoonup{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightharpoonup{\mathrm{L}^2(\mathbf{R})} 0 \ .$$

It follows that

$$u_n \longrightarrow 0 \ ,$$

but $d_n \longrightharpoonup d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Applying the div-rot lemma we arrive at *equipartition of energy*, i.e.
$u_t^2 - u_x^2 \longrightharpoonup 0$;
the kinetic and potential energy are balanced at the macroscopic level.

In order to determine the solution completely, let us take periodically modulated initial conditions (we work in spaces $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R})$ and $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$):

$$v_n(x) := \frac{1}{n} \sin(nx) \qquad \text{and} \qquad w_n(x) := \sin(nx) \ .$$

Simple calculations lead us to: $d_n(t,x) = 1 + \cos 2nx \sin 2nt \longrightharpoonup 1$,
weak $*$ in the space of Radon measures, therefore in the space of distributions as well.

Even though the sequence of solutions (u_n) weakly converges to zero, the energy density is 1, equally distributed to kinetic and potential energy.

How this can be computed in general?

Two interesting quadratic forms:

$$q(x;\mathsf{v}) := \frac{1}{2}\left[\rho(x)v_0^2 + \mathbf{A}(x)v\cdot v\right] ,$$
$$Q(x;\mathsf{v}) := \frac{1}{2}\left[\rho(x)v_0^2 - \mathbf{A}(x)v\cdot v\right] .$$

Convergence of initial data and uniformly compact support imply:

$$u_n \xrightharpoonup{*} 0 \quad \text{in} \quad \mathrm{L}^\infty(\mathbf{R};\mathrm{H}^1)\cap \mathrm{W}^{1,\infty}(\mathbf{R};\mathrm{L}^2).$$

The energy density is $d_n = q(\nabla u_n)$.

Goal: compute the distributional limit d_n, i.e. the limit

$$D_n = \int_{\langle 0,T\rangle\times\mathbf{R}^d} d_n\phi\, dtdx\ .$$

Results:

— Gilles Francfort & François Murat (1992): in linear case, C^∞ coefficients

— Patrick Gérard (1996): constant coefficients, nonlinearity with $u^p, p\leqslant 5$

— N. A. & Martin Lazar (2002): for symmetric hyperbolic systems

We have attempted to do the same for semilinear wave equation ($d=3, p=3$), with variable coefficients. The difficulties led to the study of mixed-norm Lebesgue spaces, and also prompted the introduction of H-distributions.

For nonlinear equations the L^2 theory usually does not work; one should try the L^p spaces.

◂ Back to 2nd comm lemma

A general view

We can unify the results: consider equations of the form

$$P_0(\varrho P_0 u_n) + \mathsf{P}_1 \cdot \mathbf{A}\mathsf{P}_1 u_n = 0\,,$$

where P_0 and P_1 stand for (pseudo)differential operators in time and space variables, with (principal) symbols p_0 and p_1, and $Q = \varrho p_0^2 + \mathbf{A}\mathsf{p}_1 \cdot \mathsf{p}_1$ being the symbol of the differential operator defining the left-hand side of the equation. For the H-measure $\tilde{\boldsymbol{\mu}}$ associated to $(P_0 u_n, \mathsf{P}_1 u_n)$, converging weakly in L^2 to 0, $\tilde{\boldsymbol{\mu}}$ is of the form

$$\tilde{\boldsymbol{\mu}} = \overline{\frac{\mathsf{p}\otimes\mathsf{p}}{|\mathsf{p}|^2}}\tilde{\nu}\,,$$

where $\tilde{\nu} := \mathrm{tr}\tilde{\boldsymbol{\mu}}$ is a scalar measure, and the localisation principle reads

$$Q\tilde{\nu} = 0\,.$$

Finally, the propagation principle states

$$\left\langle \frac{\xi_m \tilde{\nu}}{|\mathsf{p}|^2}, \{\phi, Q\} \right\rangle + \left\langle \frac{\nu}{|\mathsf{p}|^2}, p\,\partial_m Q \right\rangle = 0\ .$$

This covers both the classical and the parabolic case.

Good bounds: the Hörmander-Mihlin theorem

$\psi : \mathbf{R}^d \to \mathbf{C}$ is a *Fourier multiplier* on $\mathrm{L}^p(\mathbf{R}^d)$ if

$$\bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)\ , \qquad \text{for } \theta \in \mathcal{S}(\mathbf{R}^d),$$

and

$$\mathcal{S}(\mathbf{R}^d) \ni \theta \mapsto \bar{\mathcal{F}}(\psi\mathcal{F}(\theta)) \in \mathrm{L}^p(\mathbf{R}^d)$$

can be extended to a continuous mapping $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$.

Theorem. [Hörmander-Mihlin] *Let $\psi \in \mathrm{L}^\infty(\mathbf{R}^d)$ have partial derivatives of order less than or equal to $\kappa = [\frac{d}{2}] + 1$. If for some $k > 0$*

$$(\forall r > 0)(\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d) \qquad |\boldsymbol{\alpha}| \leqslant \kappa \implies \int_{\frac{r}{2}\leqslant|\boldsymbol{\xi}|\leqslant r} |\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \leqslant k^2 r^{d-2|\boldsymbol{\alpha}|}\ ,$$

then for any $p \in \langle 1, \infty\rangle$ and the associated multiplier operator $\mathcal{A}_\psi$ there exists a C_d (depending only on the dimension d) such that

$$\|\mathcal{A}_\psi\|_{\mathrm{L}^p\to\mathrm{L}^p} \leqslant C_d \max\left\{p, \frac{1}{p-1}\right\}(k + \|\psi\|_\infty)\ .$$

∎

For $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$, extended by homogeneity to $\mathbf{R}^d$, we can take $k = \|\psi\|_{\mathrm{C}^\kappa}$.

Existence of H-distributions (main theorem)

Theorem. *If* $u_n \longrightarrow 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$ *and* $v_n \xrightarrow{\ *\ } v$ *in* $\mathrm{L}^q(\mathbf{R}^d)$ *for some* $q \geqslant \max\{p', 2\}$*, then there exist subsequences* $(u_{n'})$*,* $(v_{n'})$ *and a complex valued distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$*, such that for every* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ *we have:*

$$\begin{aligned}\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} &= \lim_{n'} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})(\mathbf{x})} d\mathbf{x} \\ &= \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle \ ,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \to \mathrm{L}^p(\mathbf{R}^d)$ *is the Fourier multiplier operator with symbol* $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$*.* ∎

We call the functional μ the *H-distribution* corresponding to (a subsequence of) (u_n) and (v_n).

Of course, for $q \in \langle 1, \infty \rangle$ the weak $*$ convergence coincides with the weak convergence.

Some remarks

If (u_n), (v_n) are defined on $\Omega \subseteq \mathbf{R}^d$, extension by zero to $\mathbf{R}^d$ preserves the convergence, and we can apply the Theorem. μ is supported on $\mathrm{Cl}\,\Omega \times \mathrm{S}^{d-1}$.

In Theorem we distinguish $u_n \in \mathrm{L}^p(\mathbf{R}^d)$ and $v_n \in \mathrm{L}^q(\mathbf{R}^d)$. For $p \geqslant 2$, $p' \leqslant 2$ and we can take $q \geqslant 2$; this covers the L^2 case (including $u_n = v_n$).
The assumptions of Theorem imply that $u_n, v_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, resulting in a distribution μ of order zero (a Radon measure, not necessary bounded), instead of a more general distribution.
The real improvement in Theorem is for $p < 2$.

For applications, of interest is to extend the result to vector-valued functions. For $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

In contrast to H-measures, we cannot consider H-distributions corresponding to the same sequence, but only to a pair of sequences, and H-distribution would correspond to non-diagonal blocks for H-measures.

First commutation lemma

$\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ satisfies the conditions of the Hörmander-Mihlin theorem.
Therefore, $\mathcal{A}_\psi$ and B are bounded operators on $\mathrm{L}^r(\mathbf{R}^d)$, for any $r \in \langle 1, \infty\rangle$.
We are interested in the properties of their commutator, $C = \mathcal{A}_\psi B - B\mathcal{A}_\psi$.
If $p < r$, we can apply the classical interpolation inequality:

$$\|Cv_n\|_p \leqslant \|Cv_n\|_2^\alpha \|Cv_n\|_r^{1-\alpha} \ ,$$

for $\alpha \in \langle 0, 1\rangle$ such that $1/p = \alpha/2 + (1-\alpha)/r$.
As C is compact on $\mathrm{L}^2(\mathbf{R}^d)$ by Tartar's First commutation lemma, while it is bounded on $\mathrm{L}^r(\mathbf{R}^d)$, we get the claim.

For the most interesting case, where $p = r$, we need a better result: the Krasnosel'skij theorem (in fact, its extension to unbounded domains [N.A., M. Mišur, D. Mitrović (2018)]).

Lemma. *Assume that linear operator A is compact on $\mathrm{L}^2(\mathbf{R}^d)$ and bounded on $\mathrm{L}^r(\mathbf{R}^d)$, for some $r \in \langle 1, \infty\rangle \setminus \{2\}$. Then A is also compact on any $\mathrm{L}^p(\mathbf{R}^d)$, where $1/p = \theta/2 + (1-\theta)/r$, for a $\theta \in \langle 0, 1\rangle$.* ∎

Therefore, the commutator C is compact on all $\mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1, \infty\rangle$.

Proof of the theorem

Theorem. *If $u_n \rightharpoonup 0$ in $\mathrm{L}^p(\mathbf{R}^d)$, on a subsequnce we have:*

$$\lim_{n'\to\infty}\int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'})\overline{(\varphi_2 v_{n'})}d\mathbf{x} = \lim_{n'\to\infty}\int_{\mathbf{R}^d} (\varphi_1 u_{n'})\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})}d\mathbf{x}$$
$$= \langle \mu, \varphi_1\overline{\varphi}_2\psi\rangle .$$

The adjoint $\mathcal{A}_\psi^*$ is actually the multiplier operator $\mathcal{A}_{\bar\psi}$; this gives us the first equality (we use the sesquilinear dual product)

$${}_{\mathrm{L}^p}\Big\langle \mathcal{A}_\psi(\varphi_1 u_{n'}), \varphi_2 v_{n'}\Big\rangle_{\mathrm{L}^{p'}} = {}_{\mathrm{L}^p}\Big\langle \varphi_1 u_{n'}, \mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})\Big\rangle_{\mathrm{L}^{p'}} .$$

As $u_n \rightharpoonup 0$ in $\mathrm{L}^p(\mathbf{R}^d)$, while for a fixed $v \in \mathrm{L}^q(\mathbf{R}^d)$ we have

$$\varphi_1\overline{\mathcal{A}_{\bar\psi}(\varphi_2 v)} \in \mathrm{L}^{p'}(\mathbf{R}^d) ,$$

so by the Hörmander-Mihlin theorem it follows that

$$\lim_{n\to\infty}\int_{\mathbf{R}^d} \varphi_1 u_n\overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v)}d\mathbf{x} = 0 .$$

However, we have a product of two weakly converging sequences.

Proof of the theorem (cont.)

Write $\mathbf{R}^d = \bigcup_{l\in\mathbf{N}} K_l$, where K_l are increasing compacts; therefore $\operatorname{supp}\varphi_2 \subseteq K_l$ for some $l \in \mathbf{N}$. We have ($\chi_l := \chi_{K_l}$):

$$\begin{aligned}
\lim_{n\to\infty}\int_{\mathbf{R}^d}\varphi_1 u_n \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_n)}d\mathbf{x} &= \lim_{n\to\infty}\int_{\mathbf{R}^d}\varphi_1 u_n \overline{\mathcal{A}_{\bar\psi}[\varphi_2\chi_l(v_n - v)]}d\mathbf{x} \\
&= \lim_{n\to\infty}\int_{\mathbf{R}^d}\varphi_1\overline{\varphi}_2 u_n \overline{\mathcal{A}_{\bar\psi}(\chi_l(v_n - v))}d\mathbf{x} \\
&= \lim_{n\to\infty}\int_{\mathbf{R}^d}\varphi_1\overline{\varphi}_2 u_n \overline{\mathcal{A}_{\bar\psi}(\chi_l v_n)}d\mathbf{x}.
\end{aligned}$$

With $\varphi = \varphi_1\overline{\varphi}_2$, we have bilinear functionals:

$$\mu_{n,l}(\varphi,\psi) := \int_{\mathbf{R}^d}\varphi u_n \overline{\mathcal{A}_{\bar\psi}(\chi_l v_n)}d\mathbf{x}\ .$$

Furthermore, by the Hörmander-Mihlin theorem:

$$\left|\mu_{n,l}(\varphi,\psi)\right| \leqslant \|\varphi u_n\|_p \|\mathcal{A}_{\bar\psi}(\chi_l v_n)\|_{p'} \leqslant \tilde{C}\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}\|\varphi\|_{\mathrm{C}_0(\mathbf{R}^d)}\ ,$$

where $\tilde{C}$ depends on $\mathrm{L}^p(K_l)$, $\mathrm{L}^{p'}(K_l)$ bounds of the sequences (u_n), (v_n).

Now we need a lemma.

Lemma on bilinear forms

Lemma. *Let E, F be separable Banach spaces, (b_n) an equibounded sequence of bilinear forms on $E \times F$ (i.e. $|b_n(\varphi, \psi)| \leq C\|\varphi\|_E\|\psi\|_F$). Then there exists a subsequence (b_{n_k}) and a bilinear form b (with the same bound C) such that*

$$(\forall \varphi \in E)(\forall \psi \in F) \qquad \lim_k b_{n_k}(\varphi, \psi) = b(\varphi, \psi) .$$

∎

To each b_n we associate a bounded linear operator $B_n : E \longrightarrow F'$ by

$$_{F'}\langle\, B_n\varphi, \psi \,\rangle_F := b_n(\varphi, \psi) .$$

This defines a linear function, which is bounded:

$$\|B_n\varphi\|_{F'} = \sup_{\psi \neq 0} \frac{|b_n(\varphi, \psi)|}{\|\psi\|_F} \leq C\|\varphi\|_E .$$

Let $\mathcal{G} \subseteq E$ be a countable dense subset; for each $\varphi \in \mathcal{G}$ the sequence $(B_n\varphi)$ is bounded in F', so by the Banach theorem there is a subsequence such that

$$B_{n_1}\varphi \overset{*}{\rightharpoonup} \beta_1 =: B(\varphi) .$$

By repeating this construction, and applying the Cantor diagonal procedure we get a subsequence

$$(\forall \varphi \in \mathcal{G}) \qquad B_{n_k}\varphi \overset{*}{\rightharpoonup} B(\varphi) ,$$

such that $\|B(\varphi)\|_{F'} \leqslant C\|\varphi\|_E$.

Proof of the lemma completed

Now it is standard to extend B to a bounded linear operator on the whole space E. Clearly:

$$b(\varphi,\psi) := {}_{F'}\langle\, B\varphi, \psi\,\rangle_F = \lim_k {}_{F'}\langle\, B_{n_k}\varphi, \psi\,\rangle_F = \lim_k b_{n_k}(\varphi,\psi)\ .$$

Q.E.D.

Recall: $\mathrm{C}_{K_l}(\mathbf{R}^d) := \{\varphi \in \mathrm{C}(\mathbf{R}^d) : \operatorname{supp}\varphi \subseteq K_l\}$
Then we have:

$$\mathrm{C}_c(\mathbf{R}^d) = \bigcup_{l\in\mathbf{N}} \mathrm{C}_{K_l}(\mathbf{R}^d)\ .$$

For each $l \in \mathbf{N}$ we apply Lemma to obtain operators

$$B^l \in \mathcal{L}(\mathrm{C}_{K_l}(\mathbf{R}^d); (\mathrm{C}^\kappa(\mathrm{S}^{d-1}))')\ .$$

Furthermore, for the construction of B^l we can start with a defining subsequence for B^{l-1}, so that the convergence will remain valid on $\mathrm{C}_{K_{l-1}}(\mathbf{R}^d)$, in such a way obtaining that B^l is an extension of B^{l-1}.

Proof of the theorem completed

This allows us to define the operator B on $\mathrm{C}_c(\mathbf{R}^d)$:
for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$ we take $l \in \mathbf{N}$ such that $\operatorname{supp}\varphi \subseteq K_l$, and set $B\varphi := B^l\varphi$.

The definition is good, and we have a bounded operator in uniform norm:

$$\|B\varphi\|_{(\mathrm{C}^\kappa(\mathrm{S}^{d-1}))'} \leqslant \tilde{C}\|\varphi\|_{\mathrm{C}_0(\mathbf{R}^d)} \ .$$

It can be extended to the completion, the Banach space $\mathrm{C}_0(\mathbf{R}^d)$.
Now we can define $\mu(\varphi, \psi) := \langle B\varphi, \psi\rangle$, which satisfies the Theorem.

Indeed, restrict B to $\mathrm{C}_c^\infty(\mathbf{R}^d)$; the restriction $\tilde{B}$ remains continuous.
$(\mathrm{C}^\kappa(\mathrm{S}^{d-1}))'$ is a subspace of $\mathcal{D}'(\mathrm{S}^{d-1})$, and we have a continuous operator from $\mathrm{C}_c^\infty(\mathbf{R}^d)$ to $\mathcal{D}'(\mathrm{S}^{d-1})$, which by the Schwartz kernel theorem can be identified to a distribution from $\mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$.

However, the bounds we had indicate that we should have a better object than just a distribution, say of order no more than $\kappa = [d/2] + 1$.

(Un)fortunately, the situation is much more complicated. Just to mention that the specific examples of H-distributions that we have are all of order 0 in both variables.

A particular Nemyckij operator

Canonical choice of $\mathrm{L}^{p'}$ sequence corresponding to an L^p, $p \in \langle 1, \infty\rangle$, sequence (u_n) is given by $v_n = \Phi_p(u_n)$, where Φ_p is an operator from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ defined by $\Phi_p(u) = |u|^{p-2}u$.

Φ_p is a nonlinear Nemytskij operator, continuous from $\mathrm{L}^p(\mathbf{R}^d)$ to $\mathrm{L}^{p'}(\mathbf{R}^d)$ and additionally we have the following bound

$$\|\Phi_p(u)\|_{\mathrm{L}^{p'}(\mathbf{R}^d)} \leqslant \|u\|^{p/p'}_{\mathrm{L}^p(\mathbf{R}^d)}.$$

It maps bounded sets in $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ topology to bounded sets in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$ topology. Hence for an L^p bounded sequence (u_n), we get that $(\Phi_p(u_n))$ is weakly precompact in $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

It is continuous from $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}^d)$ to $\mathrm{L}^{p'}_{\mathrm{loc}}(\mathbf{R}^d)$.

Example: concentration

$u \in \mathrm{L}^p_c(\mathbf{R}^d)$, and define $u_n(\mathbf{x}) = n^{\frac{d}{p}} u(n(\mathbf{x}-\mathbf{z}))$ for some $\mathbf{z} \in \mathbf{R}^d$.

Simple change of variables: $\|u_n\|_{\mathrm{L}^p(\mathbf{R}^d)} = \|u\|_{\mathrm{L}^p(\mathbf{R}^d)}$ and $u_n \longrightarrow 0$ in $\mathrm{L}^p(\mathbf{R}^d)$.

Indeed, the sequence is bounded, while for $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$

$$
\begin{aligned}
\int_{\mathbf{R}^d} u_n(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} &= \int_{\mathbf{R}^d} n^{d/p} u(n(\mathbf{x}-\mathbf{z}))\varphi(\mathbf{x})d\mathbf{x} \\
&= \int_{\mathbf{R}^d} n^{d/p-d} u(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\
&= \frac{1}{n^{d/p'}} \int_{\mathbf{R}^d} u(\mathbf{y})\chi_{\operatorname{supp} u}(\mathbf{y})\varphi(\mathbf{y}/n+\mathbf{z})d\mathbf{y} \\
&\leqslant \left(\frac{\mathrm{vol}(\operatorname{supp} u)}{n^d}\right)^{1/p'} \|u\|_{\mathrm{L}^p(\mathbf{R}^d)} \max_{\mathbf{R}^d} |\varphi|.
\end{aligned}
$$

Passing to the limit, we get our claim.

The H-distribution corresponding to sequences (u_n) and $(\Phi_p(u_n))$ is given by $\delta_{\mathbf{z}} \boxtimes \nu$, where ν is a distribution on $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$ defined for $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ by

$$
\langle \nu, \psi \rangle = \int_{\mathbf{R}^d} u(\mathbf{x}) \overline{\mathcal{A}_{\bar\psi}(|u|^{p-2}u)(\mathbf{x})} d\mathbf{x}.
$$

This distribution might be a Radon measure; we do not know for sure.

Localisation principle

Theorem. *Take* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p(\mathbf{R}^d)$, $f_n \to 0$ *in* $\mathrm{W}^{-1,q}_{\mathrm{loc}}(\mathbf{R}^d)$, *for some* $q \in \langle 1, d\rangle$, *such that*

$$\operatorname{div}(\mathsf{a}(\mathbf{x})u_n(\mathbf{x})) = f_n(\mathbf{x}) \ .$$

Take an arbitrary (v_n) *bounded in* $\mathrm{L}^\infty(\mathbf{R}^d)$, *and by* μ *denote the* H*-distribution corresponding to a subsequence of* (u_n) *and* (v_n). *Then*

$$(\mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi})\mu(\mathbf{x}, \boldsymbol{\xi}) = 0$$

in the sense of distributions on $\mathbf{R}^d \times \mathrm{S}^{d-1}$, $(\mathbf{x}, \boldsymbol{\xi}) \mapsto \mathsf{a}(\mathbf{x}) \cdot \boldsymbol{\xi}$ *being the symbol of the linear PDO with* C^κ_0 *coefficients.* ∎

In order to prove the theorem, we need a particular multiplier, the so called (Marcel) Riesz potential $I_1 := \mathcal{A}_{|2\pi\boldsymbol{\xi}|^{-1}}$, and the Riesz transforms $R_j := \mathcal{A}_{\frac{\xi_j}{i|\boldsymbol{\xi}|}}$.

Note that

$$\int I_1(\phi)\partial_j g = \int (R_j\phi)g, \quad g \in \mathcal{S}(\mathbf{R}^d).$$

Using the density argument and that R_j is bounded on $\mathrm{L}^p(\mathbf{R}^d)$, we conclude $\partial_j I_1(\phi) = -R_j(\phi)$, for $\phi \in \mathrm{L}^p(\mathbf{R}^d)$.

Compactness by compensation: L^2 case

It is well known that weak convergences are ill behaved under nonlinear transformations. Only in some particular cases of compensation it is even possible to pass to the limit in a product of two weakly converging sequences.

The prototype of this compensation effect is Murat-Tartar's div-rot lemma.

For simplicity consider 2D case, (u_n^1, u_n^2) and (v_n^1, v_n^2) converging to zero weakly in $\mathrm{L}^2(\mathbf{R}^2)$, such that $(\partial_x u_n^1 + \partial_y u_n^2)$ and $(\partial_y v_n^1 - \partial_x v_n^2)$ are both contained in a compact set of $\mathrm{H}^{-1}_{loc}(\mathbf{R}^2)$ (which then implies that they converge to zero strongly in $\mathrm{H}^{-1}_{loc}(\mathbf{R}^2)$).

We can define $\mathsf{U}_n := \begin{bmatrix} \mathsf{u}_n \\ \mathsf{v}_n \end{bmatrix}$, which (on a subsequence) defines a 4×4 H-measure $\boldsymbol{\mu}$. By the localisation principle, as the above relations can be written in the form ($\mathbf{A}^1, \mathbf{A}^2$ are 4×4 constant matrices with all entries zero except $A^1_{11} = A^2_{12} = A^2_{33} = 1$ and $A^1_{34} = -1$)

$$\mathbf{A}^1 \partial_1 \mathsf{U}_n + \mathbf{A}^2 \partial_2 \mathsf{U}_n \to 0 \text{ strongly in } \mathrm{H}^{-1}_{loc}(\mathbf{R}^2)^4 ,$$

the corresponding H-measure satisfies $(\xi_1 \mathbf{A}^1 + \xi_2 \mathbf{A}^2)\boldsymbol{\mu} = \mathbf{0}$. After straightforward calculations this shows that $u_n^1 v_n^1 + u_n^2 v_n^2 \longrightarrow 0$ weak $*$ in the sense of Radon measures (and therefore in the sense of distributions as well).

What for sequences in L^p?

For the above we have used only the non-diagonal blocks $\boldsymbol{\mu}_{12} = \boldsymbol{\mu}_{21}^*$ of

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix} ,$$

corresponding to products of u_n^i and v_n^j; in fact, the calculation shows that $\mu_{12}^{11} + \mu_{12}^{22} = 0$, which gives the above result.

Assume now (u_n^1, u_n^2) and (v_n^1, v_n^2) converging to zero weakly in $\mathrm{L}^p(\mathbf{R}^2)$ and $\mathrm{L}^{p'}(\mathbf{R}^2)$, and $(\partial_1 u_n^1 + \partial_2 u_n^2)$ bounded in $\mathrm{L}^p(\mathbf{R}^2)$, while $(\partial_2 v_n^1 - \partial_1 v_n^2)$ in $\mathrm{L}^{p'}(\mathbf{R}^2)$ (thus precompact in $\mathrm{W}^{-1,p}_{loc}(\mathbf{R}^2)$, and $\mathrm{W}^{-1,p'}_{loc}(\mathbf{R}^2)$).

Then $(u_n^1 v_n^1 + u_n^2 v_n^2)$ is bounded in $\mathrm{L}^1(\mathbf{R}^2)$, so also in $\mathcal{M}_b$ (Radon measures), and by weak $*$ compactness it has a weakly converging subsequence. However, we can say more—the whole sequence converges to zero.

Denote by μ^{ij} the H-distribution corresponding to (some sub)sequences (of) (u_n^1, u_n^2) and (v_n^1, v_n^2).
Since $(\partial_1 u_n^1 + \partial_2 u_n^2)$ is bounded in $\mathrm{L}^p(\mathbf{R}^2)$, and $(\partial_2 v_n^1 - \partial_1 v_n^2)$ is bounded in $\mathrm{L}^{p'}(\mathbf{R}^2)$, they are weakly precompact, while the only possible limit is zero, so

$$\partial_1 u_n^1 + \partial_2 u_n^2 \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^p , \qquad \text{and}$$
$$\partial_2 v_n^1 - \partial_1 v_n^2 \rightharpoonup 0 \ \text{ in } \ \mathrm{L}^{p'} .$$

From the compactness of the Riesz potential I_1 mentioned above, we conclude that for $\varphi \in \mathrm{C}_c(\mathbf{R}^2)$ and $\psi \in \mathrm{C}^\kappa(\mathrm{S}^{d-1})$ the following limit holds in $\mathrm{L}^p(\mathbf{R}^2)$:

$$\mathcal{A}_{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\frac{\xi_1}{|\boldsymbol{\xi}|}}(\varphi u_n^1) + \mathcal{A}_{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\frac{\xi_2}{|\boldsymbol{\xi}|}}(\varphi u_n^2) = \mathcal{A}_{\frac{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}(\partial_1(\varphi u_n^1) + \partial_2(\varphi u_n^2)) \to 0 \,.$$

Multiplying it first by φv_n^1 and then by φv_n^2, integrating over $\mathbf{R}^2$ and passing to the limit, we conclude from the existence theorem that:

$$\xi_1 \mu^{11} + \xi_2 \mu^{21} = 0, \qquad \text{and} \qquad \xi_1 \mu^{12} + \xi_2 \mu^{22} = 0\,.$$

Next, take

$$w_n^j = \varphi \mathcal{A}_{\frac{\psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)}{|\boldsymbol{\xi}|}}(\varphi u_n^j) \in \mathrm{W}^{1,p'}(\mathbf{R}^d), \quad j = 1, 2.$$

From the last limits on the preceeding slide we get

$$\langle (\varphi v_n^1, -\varphi v_n^2), \nabla w_n^j \rangle = -\langle \mathsf{rot}\,(\varphi v_n^1, \varphi v_n^2), w_n^j \rangle \to 0 \;\; \text{as} \;\; n \to \infty,$$

for $j = 1, 2$. Rewriting it in the integral formulation, we obtain again from the existence theorem:

$$\xi_2 \mu^{11} - \xi_1 \mu^{12} = 0, \quad \xi_2 \mu^{21} - \xi_1 \mu^{22} = 0.$$

From the algebraic relations above, we can easily conclude

$$\xi_1 \left(\mu^{11} + \mu^{22}\right) = 0 \;\; \text{and} \;\; \xi_2 \left(\mu^{11} + \mu^{22}\right) = 0,$$

implying that the distribution $\mu^{11} + \mu^{22}$ is supported on the set $\{\xi_1 = 0\} \cap \{\xi_2 = 0\} \cap \mathrm{S}^1 = \emptyset$, which implies $\mu^{11} + \mu^{22} \equiv 0$.
After inserting $\psi \equiv 1$ in the definition of H-distribution, we immediately reach the conclusion.

This proof is similar to the L^2 case, but it should be noted that we had used only a non-diagonal block of 4×4 H-measure, which corresponds to the only available 2×2 H-distribution.

There is no reason to limit oneself to two dimensions; take (u_n) and (v_n) converging weakly to zero in $\mathrm{L}^p(\mathbf{R}^d)^d$ and $\mathrm{L}^{p'}(\mathbf{R}^d)^d$, and by $\boldsymbol{\mu}$ denote $d \times d$ matrix H-distribution corresponding to some chosen subsequences of (u_n) and (v_n).

Theorem. *Let (u_n) and (v_n) be vector valued sequences converging to zero weakly in $\mathrm{L}^p(\mathbf{R}^d)^d$ and $\mathrm{L}^{p'}(\mathbf{R}^d)^d$, respectively. Assume the sequence $(\mathsf{div}\, \mathsf{u}_n)$ is bounded in $\mathrm{L}^p(\mathbf{R}^d)$, and the sequence $(\mathsf{rot}\, \mathsf{v}_n)$ is bounded in $\mathrm{L}^{p'}(\mathbf{R}^d)^{d\times d}$. Then the sequence $(\mathsf{u}_n \cdot \mathsf{v}_n)$ converges to zero in the sense of distributions (or vaguely in the sense of Radon measures).* ∎

The results carry on to loc spaces as well.

An application suggested by Darko Mitrović

For scalar conservation law with discontinuous flux, the most up to date existence result for the equation

$$u_t + \operatorname{div} \mathsf{f}(t, \mathbf{x}, u) = 0$$

is obtained under the assumptions

$$\max_{\lambda \in \mathbf{R}} |\mathsf{f}(t, \mathbf{x}, \lambda)| \in L^{2+\varepsilon}(\mathbf{R}^d_+) \ .$$

Using the H-distributions, it is possible to prove an existence result for the given equation under the assumption

$$\max_{\lambda \in \mathbf{R}} |\mathsf{f}(t, \mathbf{x}, \lambda)| \in L^{1+\varepsilon}(\mathbf{R}^d_+) \ .$$

Functions of anisotropic smoothness

Let X and Y be open sets in $\mathbf{R}^d$ and $\mathbf{R}^r$ (or C^∞ manifolds), $\Omega \subseteq X \times Y$.

By $\mathrm{C}^{l,m}(\Omega)$ we denote the space of functions f on Ω, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ and $\boldsymbol{\beta} \in \mathbf{N}_0^r$, if $|\boldsymbol{\alpha}| \leqslant l$ and $|\boldsymbol{\beta}| \leqslant m$,

$$\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f = \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega) .$$

$\mathrm{C}^{l,m}(\Omega)$ becomes a Fréchet space if we define a sequence of seminorms

$$p_{K_n}^{l,m}(f) := \max_{|\boldsymbol{\alpha}| \leqslant l, |\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} f\|_{\mathrm{L}^\infty(K_n)} ,$$

where $K_n \subseteq \Omega$ are compacts, such that $\Omega = \cup_{n \in \mathbf{N}} K_n$ and $K_n \subseteq \mathrm{Int} K_{n+1}$.

For a compact set $K \subseteq \Omega$ we define a subspace of $\mathrm{C}^{l,m}(\Omega)$

$$\mathrm{C}_K^{l,m}(\Omega) := \left\{ f \in \mathrm{C}^{l,m}(\Omega) : \ \mathrm{supp}\, f \subseteq K \right\} .$$

This subspace inherits the topology from $\mathrm{C}^{l,m}(\Omega)$, which is, when considered only on the subspace, a norm topology determined by

$$\|f\|_{l,m,K} := p_K^{l,m}(f) ,$$

and $\mathrm{C}_K^{l,m}(\Omega)$ is a Banach space (it can be identified with a proper subspace of $\mathrm{C}^{l,m}(K)$). However, if $m = \infty$ (or $l = \infty$), then we shall not get a Banach space, but a Fréchet space. As in the isotropic case, an increasing sequence of seminorms that makes $\mathrm{C}_{K_n}^{l,\infty}(\Omega)$ a Fréchet space is given by $(p_{K_n}^{l,k}), k \in \mathbf{N}_0$.

Functions of anisotropic smoothness (cont.)

We can also consider the space

$$\mathrm{C}_c^{l,m}(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}_{K_n}^{l,m}(\Omega) ,$$

of all functions with compact support in $\mathrm{C}^{l,m}(\Omega)$, and equip it by a stronger topology than the one induced from $\mathrm{C}^{l,m}(\Omega)$: by the topology of *strict inductive limit*.

More precisely, it can easily be checked that

$$\mathrm{C}_{K_n}^{l,m}(\Omega) \hookrightarrow \mathrm{C}_{K_{n+1}}^{l,m}(\Omega) ,$$

the inclusion being continuous. Also, the topology induced on $\mathrm{C}_{K_n}^{l,m}(\Omega)$ by that of $\mathrm{C}_{K_{n+1}}^{l,m}(\Omega)$ coincides with the original one, and $\mathrm{C}_{K_n}^{l,m}(\Omega)$ (as a Banach space in that topology) is a closed subspace of $\mathrm{C}_{K_{n+1}}^{l,m}(\Omega)$. Then we have that the strict inductive limit topology on $\mathrm{C}_c^{l,m}(\Omega)$ induces on each $\mathrm{C}_{K_n}^{l,m}(\Omega)$ the original topology, while a subset of $\mathrm{C}_c^{l,m}(\Omega)$ is bounded if and only if it is contained in one $\mathrm{C}_{K_n}^{l,m}(\Omega)$, and bounded there. $\mathrm{C}_c^{l,m}(\Omega)$ is a barelled space.

Of course, $\mathrm{C}_c^{\infty}(\Omega) \hookrightarrow \mathrm{C}_c^{l,m}(\Omega)$ is a continuous and dense imbedding.

Distributions of anisotropic order

Definition. A *distribution of order l in $\mathbf{x}$ and order m in $\mathbf{y}$* is any linear functional on $\mathrm{C}_c^{l,m}(\Omega)$, continuous in the strict inductive limit topology. We denote the space of such functionals by $\mathcal{D}'_{l,m}(\Omega)$.

Clearly, $\mathrm{C}_c^{\infty}(\Omega) \hookrightarrow \mathrm{C}_c^{l,m}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$, with continuous and dense imbeddings, thus $\mathrm{C}_c^{l,m}(\Omega)$ is a normal space of distributions, hence its dual $\mathcal{D}'_{l,m}(\Omega)$ forms a subspace of $\mathcal{D}'(\Omega)$. If we equip it with a strong topology, it is even continuously imbedded in $\mathcal{D}'(\Omega)$.

Lemma. *Let X and Y be C^∞ manifolds. For a linear functional u on $\mathrm{C}_c^{l,m}(X \times Y)$, the following statements are equivalent*

a) $u \in \mathcal{D}'_{l,m}(X \times Y)$,

b) $(\forall K \in \mathcal{K}(X \times Y))(\exists C > 0)(\forall \Psi \in \mathrm{C}_K^{l,m}(X \times Y)) \quad |\langle u, \Psi \rangle| \leqslant C p_K^{l,m}(\Psi)$. ∎

Statement (b) of previous lemma implies:

$$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^l(X))(\forall \psi \in \mathrm{C}_L^m(Y))$$
$$|\langle u, \varphi \boxtimes \psi \rangle| \leqslant C p_K^l(\varphi) p_L^m(\psi) \, .$$

The reverse implication would have significantly greater practical use.

Tensor product of distributions

In order to better understand the properties of elements of $\mathcal{D}'_{l,m}(\Omega)$, we shall relate them to tensor products.
The first step is to consider the algebraic tensor product $\mathrm{C}_c^l(X) \boxtimes \mathrm{C}_c^m(Y)$, the vector space of all (finite) linear combinations of functions of the form $(\phi \boxtimes \psi)(\mathbf{x}, \mathbf{y}) := \phi(\mathbf{x})\psi(\mathbf{y})$. This is a vector subspace of $\mathrm{C}_c^{l,m}(X \times Y)$.

Theorem. *Let X and Y be C^∞ manifolds, $u \in \mathcal{D}'_l(X)$ and $v \in \mathcal{D}'_m(Y)$. Then*

$$\Big(\exists! w \in \mathcal{D}'_{l,m}(X \times Y)\Big)\Big(\forall \varphi \in \mathrm{C}_c^l(X)\Big)\Big(\forall \psi \in \mathrm{C}_c^m(Y)\Big) \quad \langle w, \varphi \boxtimes \psi\rangle = \langle u, \varphi\rangle\langle v, \psi\rangle.$$

Furthermore, for any $\Phi \in \mathrm{C}_c^{l,m}(X \times Y)$, function $V : \mathbf{x} \mapsto \langle v, \Phi(\mathbf{x}, \cdot)\rangle$ is in $\mathrm{C}_c^l(X)$, while $U : \mathbf{y} \mapsto \langle u, \Phi(\cdot, \mathbf{y})\rangle$ is in $\mathrm{C}_c^m(Y)$, and we have that

$$\langle w, \Phi\rangle = \langle u, V\rangle = \langle v, U\rangle.$$

∎

Simple operations

Lemma. *If* $u \in \mathcal{D}'_{l,m}(X \times Y)$ *then, for any* $\psi \in \mathrm{C}^{l,m}(X \times Y)$*,* ψu *is a well defined distribution of order at most* (l, m)*.* ∎

Theorem. *Let* $u \in \mathcal{D}'_{l,m}(X \times Y)$ *and take* $F \subseteq X \times Y$ *relatively compact set such that* $\operatorname{supp} u \subseteq F$*. Then there exists unique linear functional* $\tilde{u}$ *on* $\mathcal{Q} := \{\varphi \in \mathrm{C}^{l,m}(X \times Y) : \ F \cap \operatorname{supp} \varphi \Subset X \times Y\}$ *such that*

a) $(\forall \varphi \in \mathrm{C}_c^{l,m}(X \times Y)) \quad \langle \tilde{u}, \varphi \rangle = \langle u, \varphi \rangle$*,*

b) $(\forall \varphi \in \mathrm{C}^{l,m}(X \times Y)) \quad F \cap \operatorname{supp} \varphi = \emptyset \quad \implies \quad \langle \tilde{u}, \varphi \rangle = 0$*.*

The domain of $\tilde{u}$ *is largest for* $F = \operatorname{supp} u$*.* ∎

First conjecture

Let X, Y be C^∞ manifolds and u a linear functional on $\mathrm{C}_c^{l,m}(X \times Y)$. If $u \in \mathcal{D}'(X \times Y)$ and satisfies

$$(\forall K \in \mathcal{K}(X))(\forall L \in \mathcal{K}(Y)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^\infty(X))(\forall \psi \in \mathrm{C}_L^\infty(Y))$$
$$|\langle u, \varphi \boxtimes \psi \rangle| \leqslant C p_K^l(\varphi) p_L^m(\psi) \;,$$

then u can be uniquely extended to $\mathcal{D}'_{l,m}(X \times Y)$.

If it were true, then the H-distribution μ would belong to $\mathcal{D}'_{0,\kappa}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it would be a distribution of order 0 in $\mathbf{x}$ and of order not more than κ in $\boldsymbol{\xi}$.

Indeed, from the proof of the existence theorem, we already have $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ and the following bound with $\varphi := \varphi_1 \overline{\varphi_2}$:

$$|\langle \mu, \varphi \boxtimes \psi \rangle| \leqslant C \|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})} \|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} \;,$$

where C does not depend on φ and ψ.
It is not true!

We need a more complicated result.

Schwartz kernel theorem

Theorem. *Let X and Y be two differentiable manifolds.*

a) *Let $K \in \mathcal{D}'_{l,m}(X \times Y)$. Then for each $\varphi \in \mathrm{C}^l_c(X)$ the linear form K_φ, defined by $\psi \mapsto \langle K, \varphi \boxtimes \psi \rangle$, is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ with its inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.*

b) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator, in the pair of topologies as above. Then there exists unique distribution $K \in \mathcal{D}'(X \times Y)$ such that for any $\varphi \in \mathrm{C}^\infty_c(X)$ and $\psi \in \mathrm{C}^\infty_c(Y)$*

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle.$$

Furthermore, $K \in \mathcal{D}'_{l,d(m+2)}(X \times Y)$. ∎

Different strategies of proof

- ◦ regularisation? (Schwartz)
- ◦ constructive proof? (Simanca, Gask, Ehrenpreis)
- ◦ nuclear spaces? (Trèves)
- ◦ structure theorem, on manifolds (Dieudonné)

From kernel to operator (a)

$\varphi \in \mathrm{C}_c^l(X)$; prove the continuity of K_φ on $\mathrm{C}_c^m(Y)$ (it is clearly linear since the tensor product is bilinear, while K is linear).

i.e. for $H \in \mathcal{K}(Y)$, mapping $\psi \mapsto \langle K_\varphi, \psi \rangle$ is a cont. lin. funct. on $\mathrm{C}_H^m(Y)$.

We can assume X and Y to be open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$.
Indeed, first take an open covering of Y, consisting of chart domains, and a partition of unity (f_α) subordinate to that covering such that $\sum_\alpha f_\alpha(\mathbf{y}) = 1, \mathbf{y} \in H$ (note that the sum is finite).
Similarly for φ, thus limiting ourselves to domains of a pair of charts.

By [Gösser, Kunzinger & al., Chapter 3.1.4], we can identify distributions localised on chart domains with distributions on subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$. Thus, in what follows we shall assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$.

We shall therefore show that there exists a constant $C > 0$ such that for any $\psi \in \mathrm{C}_H^m(Y)$ it holds

$$|\langle K_\varphi, \psi \rangle| \leqslant C \max_{|\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\beta}} \psi\|_{\mathrm{L}^\infty(H)} ,$$

for m finite, while for $m = \infty$ we should modify the above to

$$(\exists\, m' \in \mathbf{N})(\exists\, C > 0)(\forall\, \psi \in \mathrm{C}_H^\infty(Y)) \quad |\langle K_\varphi, \psi \rangle| \leqslant C \max_{|\boldsymbol{\beta}| \leqslant m'} \|\partial^{\boldsymbol{\beta}} \psi\|_{\mathrm{L}^\infty(H)} .$$

From kernel to operator (a)(cont.)

K is a distribution of anisotropic order on $X \times Y$:

$$(\forall\, M \in \mathcal{K}(X \times Y))(\exists \tilde{C} > 0)(\forall \Psi \in \mathrm{C}_c^{l,m}(X \times Y))$$
$$\operatorname{supp}\Psi \subseteq M \Longrightarrow |\langle K, \Psi \rangle| \leqslant \tilde{C} \max_{|\boldsymbol{\alpha}| \leqslant l, |\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\alpha},\boldsymbol{\beta}} \Psi\|_{\mathrm{L}^\infty(M)} ,$$

with obvious modifications if either l or m is infinite,

by taking M to be of the form $L \times H$, with $L \in \mathcal{K}(X)$, and $\Psi = \varphi \boxtimes \psi$ such that $\operatorname{supp}\varphi \subseteq L$, we have

$$|\langle K_\varphi, \psi \rangle| = |\langle K, \varphi \boxtimes \psi \rangle| \leqslant \tilde{C} \max_{|\boldsymbol{\alpha}| \leqslant l, |\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\alpha}} \varphi \boxtimes \partial^{\boldsymbol{\beta}} \psi\|_{\mathrm{L}^\infty(L \times H)}$$
$$\leqslant \tilde{C} \max_{|\boldsymbol{\alpha}| \leqslant l} \|\partial^{\boldsymbol{\alpha}} \varphi\|_{\mathrm{L}^\infty(L)} \max_{|\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\beta}} \psi\|_{\mathrm{L}^\infty(H)} \leqslant C \max_{|\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\beta}} \psi\|_{\mathrm{L}^\infty(H)} ,$$

and therefore $K_\varphi \in \mathcal{D}'_m(Y)$.

From kernel to operator (a)(cont.)

The linearity of mapping $\varphi \mapsto K_\varphi$ readily follows from the bilinearity of tensor product and the linearity of K.

For continuity, take an arbitrary $L \in \mathcal{K}(X)$ and an arbitrary $\psi \in \mathrm{C}_c^m(Y)$. We need to show the existence of $\bar{C} > 0$ such that

$$|\langle K_\varphi, \psi \rangle| \leqslant \bar{C} \max_{|\boldsymbol{\alpha}| \leqslant l} \|\partial^{\boldsymbol{\alpha}} \varphi\|_{\mathrm{L}^\infty(L)} .$$

However, we have already shown that above: just take

$$\bar{C} = \tilde{C} \max_{|\boldsymbol{\beta}| \leqslant m} \|\partial^{\boldsymbol{\beta}} \psi\|_{\mathrm{L}^\infty(H)} .$$

Therefore, the mapping $\varphi \mapsto K_\varphi$, from $\mathrm{C}_c^l(X)$ to $\mathcal{D}'_m(Y)$ is linear and continuous.

From operator to kernel (b): uniqueness and overview

Let us first prove the uniqueness. By formula

$$\langle K, \varphi \boxtimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle \ ,$$

a continuous functional K on $\mathrm{C}_c^\infty(X) \boxtimes \mathrm{C}_c^\infty(Y)$ is defined. As it is defined on a dense subset of $\mathrm{C}_c^\infty(X \times Y)$, such K is uniquely determined on the whole $\mathrm{C}_c^\infty(X \times Y)$.

The proof of existence will be divided into two steps. In the first step we assume that X and Y are open subsets of $\mathbf{R}^d$ and $\mathbf{R}^r$, and additionally, that the range of A is $\mathrm{C}(Y) \subseteq \mathcal{D}'_m(Y)$ (understood as distributions which can be identified with continuous functions). This will allow us to write explicitly the action of $A\varphi$ on a test function $\psi \in \mathrm{C}_c^m(Y)$, which will finally enable us to define the kernel K. In the second step, we shall use a partition of unity and the structure theorem of distributions to reduce the problem to the first step.

From operator to kernel (b): existence under additional assumptions

Additionally assume that X and Y are open and bounded subsets of euclidean spaces, and that for each $\varphi \in \mathrm{C}_c^l(X)$, $A\varphi \in \mathrm{C}(Y)$.

Its action on a test function $\psi \in \mathrm{C}_c^m(Y)$ is given by

$$\langle A\varphi, \psi \rangle = \int_Y (A\varphi)(\mathbf{y})\psi(\mathbf{y})d\mathbf{y} \; .$$

Continuity of A implies that $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$ is continuous when the range is equipped with the weak $*$ topology inherited from $\mathcal{D}'_m(Y)$.

As the latter is a Hausdorff space, that operator has a closed graph, but this remains true even when we replace the topology on $\mathrm{C}(Y)$ by its standard Fréchet topology [Narici & Beckenstein, Exercise 14.101(a)], which is stronger.

Now we can apply the Closed graph theorem [Narici & Beckenstein, Theorem 14.3.4(b)], as $\mathrm{C}_c^l(X)$ is barreled, as a strict inductive limit of barreled spaces, to conclude that $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$ is continuous with usual strong topologies on its domain and range.

(b): existence under additional assumptions (cont.)

For $\mathbf{y} \in Y$ consider a linear functional $F_{\mathbf{y}} : \mathrm{C}^l_c(X) \longrightarrow \mathbf{C}$ defined by

$$F_{\mathbf{y}}(\varphi) = (A\varphi)(\mathbf{y}) \ .$$

Since $A\varphi$ is a continuous function, $F_{\mathbf{y}}$ is well-defined and continuous as a composition of continuous mappings, thus a distribution in $\mathcal{D}'_l(X)$.

Take a test function $\Psi \in \mathrm{C}^{l,0}_c(X \times Y)$, and fix its second variable (get a function from $\mathrm{C}^l_c(X)$) and apply $F_{\mathbf{y}}$; we are interested in the properties of this mapping:

$$\mathbf{y} \mapsto F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) = \Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y}) \ .$$

Clearly, it is well defined on Y, with a compact support contained in the projection $\pi_Y(\operatorname{supp}\Psi)$. Furthermore, we have:

$$\begin{aligned}\Big|F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\Big| = \Big|\Big(A\Psi(\cdot, \mathbf{y})\Big)(\mathbf{y})\Big| &\leqslant \|A\Psi(\cdot, \mathbf{y})\|_{\mathrm{L}^\infty(\pi_Y(\operatorname{supp}\Psi))} \\ &\leqslant C\|\Psi(\cdot, \mathbf{y})\|_{\mathrm{C}^l_{\pi_Y(\operatorname{supp}\Psi)}(X)} \leqslant C\|\Psi\|_{\mathrm{C}^{l,0}_{\operatorname{supp}\Psi}(X \times Y)} \ .\end{aligned}$$

(b): existence under additional assumptions (cont.)

We show sequential continuity: take a sequence $\mathbf{y}_n \to \mathbf{y}$ in Y. Denote $H = \pi_X(\operatorname{supp}\Psi)$ and let $L \subseteq Y$ be a compact such that $\mathbf{y}_n, \mathbf{y} \in L$; Ψ is uniformly continuous on compact $H \times L$.

This is also valid for $\partial_{\mathbf{x}}^{\boldsymbol{\alpha}}\Psi$, where $|\boldsymbol{\alpha}| \leqslant l$, thus $\Psi(\cdot, \mathbf{y}_n) \longrightarrow \Psi(\cdot, \mathbf{y})$ in $\mathrm{C}_c^l(X)$.

As A is continuous, the convergence is carried to $\mathrm{C}(Y)$, i.e. to uniform convergence on compacts of a sequence of functions $A\Psi(\cdot, \mathbf{y}_n)$ to $A\Psi(\cdot, \mathbf{y})$. In particular,$(A\Psi(\cdot, \mathbf{y}_n))(\bar{\mathbf{y}}) - (A\Psi(\cdot, \mathbf{y}))(\bar{\mathbf{y}})$ is arbitrary small independently of $\bar{\mathbf{y}} \in L$, for large enough n.

On the other hand, $A\Psi(\cdot, \mathbf{y})$ is uniformly continuous, thus $(A\Psi(\cdot, \mathbf{y}))(\bar{\mathbf{y}}) - (A\Psi(\cdot, \mathbf{y}))(\mathbf{y})$ is small for large n, independetly of $\bar{\mathbf{y}} \in L$. In other terms, we have the required convergence

$$F_{\mathbf{y}_n}(\Psi(\cdot, \mathbf{y}_n)) \longrightarrow F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y})) \ .$$

A continuous function with compact support is summable, so we can define K on $\mathrm{C}_c^{l,0}(X \times Y)$:

$$\langle K, \Psi \rangle = \int_Y F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\, d\mathbf{y} \ ,$$

which is obviously linear in Ψ, as $F_{\mathbf{y}}$ is.

(b): existence under additional assumptions (cont.)

For continuity of K, we cannot follow [Dieudonné, 23.9.2], as our spaces are not Montel.

However, we can check that K is continuous at zero (modifications for $l = \infty$):

$$(\forall\, H \in \mathcal{K}(X))(\forall\, L \in \mathcal{K}(Y))(\exists\, C > 0)(\forall\, \Psi \in \mathrm{C}_c^{l,0}(X \times Y))$$
$$\operatorname{supp} \Psi \subseteq H \times L \quad \Longrightarrow \quad |\langle K, \Psi \rangle| \leqslant C \|\Psi\|_{\mathrm{C}^{l,0}_{K\times L}(X\times Y)}\ .$$

The continuity of $A : \mathrm{C}_c^l(X) \longrightarrow \mathrm{C}(Y)$, for Ψ supported in $H \times L$ and the fact that the support of $A\Psi(\cdot, \mathbf{y})$ is contained in L gives us the estimate

$$\left| \int_Y F_{\mathbf{y}}(\Psi(\cdot, \mathbf{y}))\, d\mathbf{y} \right| \leqslant (\mathrm{vol} L) C \|\Psi\|_{\mathrm{C}^{l,0}_{K\times L}(X\times Y)}\ ,$$

as needed.

Finally, it is easy to check that for $\varphi \in \mathrm{C}_c^\infty(X)$ and $\psi \in \mathrm{C}_c^\infty(Y)$, we have:

$$\begin{aligned}\langle K, \varphi \boxtimes \psi \rangle = \int_Y F_{\mathbf{y}}(\varphi \boxtimes \psi(\mathbf{y})) d\mathbf{y} &= \int_Y F_{\mathbf{y}}(\varphi)\psi(\mathbf{y}) d\mathbf{y} \\ &= \int_Y (A\varphi)(\mathbf{y})\psi(\mathbf{y}) d\mathbf{y} = \langle A\varphi, \psi \rangle\ .\end{aligned}$$

(b) existence in general: reduction to charts

Let (U_α) and (V_β) be covers consisting of relatively compact open sets.

It is sufficient to show existence of distributions $K_{\alpha\beta}$ on $U_\alpha \times V_\beta$, which satisfy

$$\langle A\varphi, \psi \rangle = \langle K_{\alpha\beta}, \varphi \boxtimes \psi \rangle , \quad \varphi \in \mathrm{C}_c^\infty(U_\alpha), \psi \in \mathrm{C}_c^\infty(V_\beta) .$$

Indeed, the uniqueness of $K \in \mathcal{D}'(X \times Y)$ then follows from the fact that two distributions $K_{\alpha\beta}$ and $K_{\gamma\delta}$ will coincide on open sets $(U_\alpha \cap U\gamma) \times (V_\beta \cap V_\delta)$ of $X \times Y$, while the existence of K will be a result of the localisation theorem [Dieudonné, 17.4.2].

Furthermore, if we assume that U_α and V_β lie within domains of some charts of X and Y, in the light of results of [Gösser, Kunzinger & al., Chapter 3.1.4], we can identify the distributions localised to these chart domains with distributions on open subsets of $\mathbf{R}^d$. Thus, without loss of generality, we assume that U and V are relatively compact open subsets of $\mathbf{R}^d$.

(b) existence in general: the structure theorem

Consider $\tilde{A} : \mathrm{C}^l_c(U) \to \mathcal{D}'_m(V)$ defined by: for $\varphi \in \mathrm{C}^l_c(U)$ and $\psi \in \mathrm{C}^m_c(V)$

$$\langle \tilde{A}\varphi, \psi \rangle = \langle A\varphi, \psi \rangle \ .$$

$\tilde{A}$ is well-defined, and by the assumptions continuous.

Take a relatively compact open neighbourhood W of $\mathrm{Cl}\, V$ in Y and pick a smooth cut-off function ρ being one on $\mathrm{Cl}\, V$ and supported in W.

For $\varphi \in \mathrm{C}^l_c(U)$, $\rho\tilde{A}\varphi \in \mathcal{D}'_m(W)$ and has a compact support. Next we use the Structure theorem for distributions: from its proof [Friedlander & Joshi, Theorem 5.4.1], we can write

$$\rho\tilde{A}\varphi = \left(\partial_1^{m+2} \dots \partial_d^{m+2}\right) \Big(E_{m+2} * (\rho\tilde{A}\varphi) \Big) \ ,$$

where E_{m+2} is the fundamental solution of $\partial_1^{m+2} \dots \partial_d^{m+2}$ (derivatives in $\mathbf{y}$), i.e. it satisfies the equation $\left(\partial_1^{m+2} \dots \partial_d^{m+2}\right) E_{m+2} = \delta_0$ (explicit formula for E_{m+2} in loc.cit.), and $E_{m+2} * (\rho\tilde{A}\varphi)$ is a continuous function.

Denoting by $\widetilde{E}_{m+2}*$ the transpose of $E_{m+2}*$, for $\varphi \in \mathrm{C}^l_c(U)$ and $\psi \in \mathrm{C}^m_c(W)$

$$\Big\langle E_{m+2} * (\rho\tilde{A}\varphi), \psi \Big\rangle = \Big\langle \tilde{A}\varphi, \rho\widetilde{E}_{m+2} * \psi \Big\rangle \ ,$$

concluding that $\varphi \mapsto E_{m+2} * (\rho\tilde{A}\varphi)$ is continuous from $\mathrm{C}^l_c(U)$ to $\mathcal{D}'_m(W)$.

(b) existence in general: reduction to special case

Now we can find $R \in \mathcal{D}'_{l,0}(U \times W)$ such that for all $\varphi \in \mathrm{C}_c^\infty(U)$ and $\psi \in \mathrm{C}_c^\infty(W)$ it holds

$$\left\langle E_{m+2} * (\rho \tilde{A} \varphi), \psi \right\rangle = \langle R, \varphi \boxtimes \psi \rangle \, .$$

Taking $\varphi \in \mathrm{C}_c^\infty(U)$ and $\psi \in \mathrm{C}_c^\infty(V)$, we have

$$\begin{aligned}
\left\langle R, \varphi \boxtimes \left(\partial_1^{m+2} \ldots \partial_d^{m+2}\right) \psi \right\rangle &= \left\langle E_{m+2} * (\rho \tilde{A} \varphi), \left(\partial_1^{m+2} \ldots \partial_d^{m+2}\right) \psi \right\rangle \\
&= (-1)^{d(m+2)} \left\langle \left(\partial_1^{m+2} \ldots \partial_d^{m+2}\right) \left(E_{m+2} * (\rho \tilde{A} \varphi)\right), \psi \right\rangle \\
&= (-1)^{d(m+2)} \left\langle \rho \tilde{A} \varphi, \psi \right\rangle \\
&= (-1)^{d(m+2)} \left\langle \tilde{A} \varphi, \rho \psi \right\rangle \\
&= (-1)^{d(m+2)} \langle A \varphi, \psi \rangle,
\end{aligned}$$

which gives $\langle A\varphi, \psi \rangle = (-1)^{d(m+2)} \left\langle \left(\partial_1^{m+2} \ldots \partial_d^{m+2}\right) R, \varphi \boxtimes \psi \right\rangle$, where the derivatives are taken with respect to the variable $\mathbf{y}$. Since R was an element of $\mathcal{D}'_{l,0}(U \times W)$, we conclude that $A \in \mathcal{D}'_{l,d(m+2)}(U \times V)$.

Remarks

Note that in part (b) we did not get $K \in \mathcal{D}'_{l,m}(X \times Y)$, as one would expect. The order with respect to $\mathbf{x}$ variable remained the same, but the order with respect to $\mathbf{y}$ increased from m to $d(m+2)$. Interchanging the roles of X and Y, the same proof gives $K \in \mathcal{D}'_{d(l+2),m}(X \times Y)$, where order with respect to $\mathbf{y}$ remained the same, but order with respect to the $\mathbf{x}$ variable increased from l to $d(l+2)$. Since uniqueness of $K \in \mathcal{D}'(X \times Y)$ has already been determined, we conclude that $K \in \mathcal{D}'_{l,d(m+2)}(X \times Y) \cap \mathcal{D}'_{d(l+2),m}(X \times Y)$. It might be interesting to see some additional properties of that intersection.

If one used a more constructive proof of the Schwartz kernel theorem, for example [Simanca, Theorem 1.3.4], one would end up increasing the order with respect to both variables $\mathbf{x}$ and $\mathbf{y}$. This occurs naturally, because one needs to secure the integrability of the function which is used to define the kernel function.

One interesting approach to the kernel theorem is given in [Trèves, Chapter 51]. This approach is based on deep results of functional analysis on tensor products of nuclear spaces of Alexander Grothendieck. This approach might result in further improvements of the preceeding theorem. This is a subject of our current ongoing research.

Consequence for H-distributions

By the previous theorem the H-distribution μ mentioned at the beginning belongs to the space $\mathcal{D}'_{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$, i.e. it is a distribution of order 0 in $\mathbf{x}$ and of order not more than $d(\kappa+2)$ in $\boldsymbol{\xi}$.

Indeed, we already have $\mu \in \mathcal{D}'(\mathbf{R}^d \times \mathrm{S}^{d-1})$ and the following bound with $\varphi := \varphi_1\overline{\varphi_2}$:

$$|\langle \mu, \varphi \boxtimes \psi \rangle| \leqslant C\|\psi\|_{\mathrm{C}^\kappa(\mathrm{S}^{d-1})}\|\varphi\|_{\mathrm{C}_{K_l}(\mathbf{R}^d)} ,$$

where C does not depend on φ and ψ.

Now we just need to apply the Schwartz kernel theorem given above to conclude that μ is a continuous linear functional on $\mathrm{C}_c^{0,d(\kappa+2)}(\mathbf{R}^d \times \mathrm{S}^{d-1})$.

⏩ To mixed-norm case

Thank you for your attention!

Lebesgue spaces with mixed norm

For $\mathbf{p} \in [1, \infty\rangle^d$, by $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ denote the space of f on $\mathbf{R}^d$ with finite norm

$$\|f\|_{\mathbf{p}} = \left(\int_{\mathbf{R}} \cdots \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}} |f(x_1, \ldots, x_d)|^{p_1} dx_1 \right)^{p_2/p_1} dx_2 \right)^{p_3/p_2} \cdots dx_d \right)^{1/p_d} ,$$

and analogously for $p_i = \infty$.
These Banach spaces can be seen as vector-valued Lebesgue spaces in the sense

$$\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) = \mathrm{L}^{p_d}_{x_d}(\mathbf{R}; \mathrm{L}^{(p_1, \ldots, p_{d-1})}_{x_1, \ldots, x_{d-1}}(\mathbf{R}^{d-1})) .$$
$$\mathbf{p}' = (p_1', \ldots, p_d'), \quad \frac{1}{p_i} + \frac{1}{p_i'} = 1$$

Some facts:

(a) $\mathcal{S} \hookrightarrow \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$,
(b) $\mathcal{S}$ is dense in $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, for $\mathbf{p} \in [1, \infty\rangle^d$,
(c) $\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)$ is topological dual of $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, for $\mathbf{p} \in [1, \infty\rangle^d$,
(d) $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \hookrightarrow \mathcal{S}'$.

Basic results

Some generalisations of classical results are still valid:

dominated convergence for $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ spaces, $\mathbf{p} \in [1, \infty\rangle^d$
Let (f_n) be sequence of measurable functions. If $f_n \longrightarrow f$ (ss), and if there exists $G \in \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ such that $|f_n| \leqslant G$ (ss), for $n \in \mathbf{N}$, then $\|f_n - f\|_{\mathbf{p}} \longrightarrow 0$.

Minkowski ineaquality for integrals
For $\mathbf{p} \in [1, \infty]^{d_1}$ and $f \in \mathrm{L}^{(\mathbf{p},1,\ldots,1)}(\mathbf{R}^{d_1+d_2})$ we have

$$\left\|\int_{\mathbf{R}^{d_2}} f(\mathbf{x}, \mathbf{y}) d\mathbf{y}\right\|_{\mathbf{p}} \leqslant \int_{\mathbf{R}^{d_2}} \|f(\cdot, \mathbf{y})\|_{\mathbf{p}} d\mathbf{y}.$$

Hölder's ineaquality and its converse
For $\mathbf{p} \in [1, \infty]^d$ we have

$$\left|\int_{\mathbf{R}^d} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}\right| \leqslant \|f\|_{\mathbf{p}} \|g\|_{\mathbf{p}'}.$$

and

$$\|f\|_{\mathbf{p}} = \sup_{g \in \mathrm{S}_{\mathbf{p}'}} \left|\int f \bar{g} d\mathbf{x}\right| = \sup_{g \in \mathrm{S}_{\mathbf{p}'} \cap \mathcal{S}} \left|\int f \bar{g} d\mathbf{x}\right|,$$

where $\mathrm{S}_{\mathbf{p}'}$ is a unit sphere in $\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)$.

Boundedness of pseudodifferential operators on classical spaces

(ρ, δ)-*symbol* of order $m \in \mathbf{N}$ $\quad (\lambda(\boldsymbol{\xi}) = \sqrt{1 + 4\pi^2 |\boldsymbol{\xi}|^2})$

$$|\partial_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} a(\mathbf{x}, \boldsymbol{\xi})| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} \lambda^{m - \rho|\boldsymbol{\beta}| + \delta|\boldsymbol{\alpha}|}(\boldsymbol{\xi}),$$

and the associated operator $a(\cdot, D) : \mathcal{S} \longrightarrow \mathcal{S}$

$$(a(\mathbf{x}, D)\varphi)(\mathbf{x}) = \int_{\mathbf{R}^d} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} a(\mathbf{x}, \boldsymbol{\xi}) \hat{\varphi}(\boldsymbol{\xi}) \, d\boldsymbol{\xi}.$$

Using the adjoint operator, it can be extended to an operator on $\mathcal{S}'$.

Classical boundedness results on Lebesgue spaces:

- L. Hörmander: for $0 \leqslant \delta \leqslant \rho \leqslant 1$ and $\delta < 1$ the necessary condition is

$$m \leqslant -d(1 - \rho) \left| \frac{1}{2} - \frac{1}{p} \right|.$$

- C. Fefferman: for $1 < p < \infty$ this condition is also sufficient.

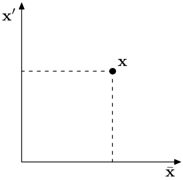
The strongest results are for $\rho = 1$, valid even for $m = 0$, which easily leads to generalisations for Sobolev spaces ($|\boldsymbol{\gamma}| \leqslant k - m$) via

$$(\partial^{\boldsymbol{\gamma}} a)(\cdot, D) = \sum_{|\boldsymbol{\alpha}| \leqslant k} a_{\boldsymbol{\alpha}}(\cdot, D) \partial^{\boldsymbol{\alpha}} .$$

Notation

$$\mathbf{x} = (\bar{\mathbf{x}}, \mathbf{x}'),\ \bar{\mathbf{x}} = (x_1, \ldots, x_r),\ \mathbf{x}' = (x_{r+1}, \ldots, x_d),\ 0 \leqslant r \leqslant d-1,$$

$$\mathrm{L}^{\bar{\mathbf{p}},\,p}(\mathbf{R}^d) = \mathrm{L}^{(\bar{\mathbf{p}},p,\ldots,p)}(\mathbf{R}^d),\ \|f\|_{\bar{\mathbf{p}},\,p} = \|f\|_{(\bar{\mathbf{p}},\,p,\ldots,\,p)},\ \bar{\mathbf{p}} = (p_1, \ldots, p_r).$$



If $r = 0$: $\quad \|f(\cdot, \mathbf{x}')\|_{\bar{\mathbf{p}}} = |f(\mathbf{x}')|, \quad \|f\|_{\bar{\mathbf{p}},\,p} = \|f\|_{\mathrm{L}^p}$.

Distribution function:

$$\lambda_f(\alpha) = \lambda(f;\alpha) = \mathrm{vol}\{\mathbf{x} \in \mathbf{R}^d : |f(\mathbf{x})| > \alpha\}.$$

(a) λ_f is non-increasing and right continuous.
(b) If $|f| \leqslant |g|$, then $\lambda_f \leqslant \lambda_g$.
(c) If $|f_n| \nearrow |f|$, then $\lambda_{f_n} \nearrow \lambda_f$.
(d) If $f = g + h$, it follows $\lambda(f;\alpha) \leqslant \lambda(g; \frac{\alpha}{2}) + \lambda(h; \frac{\alpha}{2})$.

General framework

Theorem. *Assume:*

1) $A, A^* : \mathrm{L}^\infty_c(\mathbf{R}^d) \to \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$ *are formally adjoint linear operators.*
2) *For both* $T = A$ *and* $T = A^*$ *there exist constants* $N > 1$ *and* $c_1 > 0$ *satisfying*

$$(\forall\, r \in 0..(d{-}1))(\forall\, \mathbf{x}'_0 \in \mathbf{R}^{d-r})(\forall\, t > 0) \quad \int\limits_{|\mathbf{x}'-\mathbf{x}'_0|_\infty > Nt} \|Tf(\cdot, \mathbf{x}')\|_{\bar{\mathbf{p}}}\, d\mathbf{x}' \leqslant c_1 \|f\|_{\bar{\mathbf{p}},1}\ ,$$

for any function f *in a subspace of* $\mathrm{L}^\infty_c(\mathbf{R}^d)$ *determined by properties:*
(a) $\operatorname{supp} f \subseteq \mathbf{R}^r \times \{\mathbf{x}' : |\mathbf{x}' - \mathbf{x}'_0|_\infty \leqslant t\}$,
(b) $\int_{\mathbf{R}^{d-r}} f(\bar{\mathbf{x}}, \mathbf{x}')\, d\mathbf{x}' = 0 \quad (a.e.\ \bar{\mathbf{x}} \in \mathbf{R}^r)$.

3) *For some* $q \in \langle 1, \infty \rangle$ A *has a continuous extension to an operator from* $\mathrm{L}^q(\mathbf{R}^d)$ *to itself with norm* c_q*.*

Then A *has a continuous extension to an operator from* $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ *to itself for any* $\mathbf{p} \in \langle 1, \infty \rangle^d$*, with the norm*

$$\|A\|_{\mathrm{L}^{\mathbf{p}} \to \mathrm{L}^{\mathbf{p}}} \leqslant \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max(p_{d-j}, (p_{d-j} - 1)^{-1/p_{d-j}})(c_1 + c_q)$$

$$\leqslant c' \prod_{j=0}^{d-1} \max(p_{d-j}, (p_{d-j} - 1)^{-1/p_{d-j}})(c_1 + c_q),$$

where c *and* c' *are constants depending only on* N *and* d*.* ∎

A few words about the proof

Note that we are using $\mathrm{L}_c^\infty(\mathbf{R}^d)$ as a dense subspace, and not $\mathrm{C}_c^\infty(\mathbf{R}^d)$, as we have to use the Calderón-Zygmund decomposition.

The proof follows by repeated application of

Lemma. *Assume that* $A, A^* : \mathrm{L}_c^\infty(\mathbf{R}^d) \to \mathrm{L}_{\mathrm{loc}}^1(\mathbf{R}^d)$ *are linear operators satisfying assumptions of the theorem.*
If operator A *has a continuous extension from* $\mathrm{L}^{\bar{\mathbf{p}},\,q}(\mathbf{R}^d)$ *to itself with norm* c_q*, for some* $\bar{\mathbf{p}} \in \langle 1, \infty\rangle^r$ *and* $q \in \langle 1, \infty\rangle$*, then* A *has a continuous extension from* $\mathrm{L}^{\bar{\mathbf{p}},\,p}(\mathbf{R}^d)$ *to itself for all* $p \in \langle 1, \infty\rangle$*, with norm*

$$\|A\| \leqslant c \cdot \max(p, (p-1)^{-1/p})(c_1 + c_q),$$

where c *is a constant depending only on* N *and* d*.* ∎

If some of the consecutive p_i-s are equal, we can get a bit better estimate.

The boundedness

Teorem. *Pseudodifferential operators of class* $S^0_{1,\delta}$, $\delta \in [0,1\rangle$ *are bounded on* $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in \langle 1, \infty\rangle^d$, *with an estimate as in the previous theorem.* ∎

We have considered several venues for the proof:

- Using the techniques from N.A.& I. Ivec (2016) . . . work only for compactly supported operators.
- Modifying the apporach in M. W. Wong's book (1999), as it was done in J. Aleksić, S. Pilipović & I. Vojnović (2017) . . . in the mixed norm case some calculations did not work out.
- We followed Stein (1993): the representation of pseudodifferential operator

$$(a(\mathbf{x}, D)\varphi)(\mathbf{x}) = k(\mathbf{x}, \cdot) * \varphi \ ,$$

where the kernel $k(\mathbf{x}, \cdot)$ is a tempered distrubution such that $\widehat{k(\mathbf{x}, \cdot)} = a(\mathbf{x}, \cdot)$. Outside the origin kernel $k(\mathbf{x}, \cdot)$ is in fact a smooth function decreasing at infinity; more precisely, the following theorem holds:

Teorem. *If* $a \in S^m_{1,0}$, *then* $k \in \mathrm{C}^\infty(\mathbf{R}^d \times \mathbf{R}^d \backslash \{\mathbf{0}\})$

$$|\partial_{\boldsymbol{\alpha}} \partial^{\boldsymbol{\beta}} k(\mathbf{x}, \mathbf{z})| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta},N} |\mathbf{z}|^{-d-m-|\boldsymbol{\beta}|-N}, \qquad \mathbf{z} \neq \mathbf{0},$$

for multiindices $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$, *and* $N \in \mathbf{N}_0$ *such that* $d + m + |\boldsymbol{\beta}| + N > 0$. ∎

The key lemma

The theorem is true also for $0 \leqslant \delta < 1$, if $\boldsymbol{\alpha} = 0$!

Lemma. *For each* $N > d$ *there is a* $c > 0$ *such that for* $\bar{\mathbf{p}} \in \langle 1, \infty\rangle^r$ *and any* $f \in \bigcup_{1<p< \ infty} \mathrm{L}^p(\mathbf{R}^d)$ *satisfying:*

$\operatorname{supp} f \subseteq \mathbf{R}^r \times \{\mathbf{x}' : |\mathbf{x}' - \mathbf{x}'_0|_\infty \leqslant t\},\ \mathbf{x}'_0 \in \mathbf{R}^{d-r},\ t > 0,$

$\int f(\bar{\mathbf{x}}, \mathbf{x}') d\mathbf{x}' = 0$ *a.e.* $\bar{\mathbf{x}}$,

it holds

$$\int_{|\mathbf{x}'-\mathbf{x}'_0|_\infty > Nt} \|a(\cdot, D)f\|_{\bar{\mathbf{p}}} \mathbf{x}' \leqslant c\|f\|_{\bar{\mathbf{p}},1}.$$

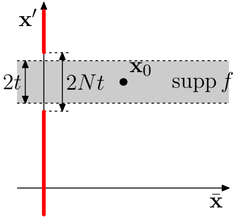


Figure 1. The support of function and the area of integration are disjoint

∎

Corollary. $a \in S^m_{1,\delta}$, $k \geqslant m$, *then* $a(\cdot, D) : \mathrm{W}^{k,\mathbf{p}}(\mathbf{R}^d) \longrightarrow \mathrm{W}^{k-m,\mathbf{p}}(\mathbf{R}^d)$ *is bounded.*

∎

Boundedness of integral operators

Another application of the general theorem on

$$Af(\mathbf{x}) = \int_{\mathbf{R}^d} K(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y},$$

and it is a known fact that they are bounded on $\mathrm{L}^p(\mathbf{R}^d)$ for $p \in [1,\infty]$ (the Schur test) if the following sufficient conditions are satisfied:

$$(\exists\, C_1, C_2 > 0)\ \int_{\mathbf{R}^d} |K(\mathbf{x},\mathbf{y})|\, d\mathbf{x} < C_1 \ \ (\text{a.e. } \mathbf{y}), \quad \int_{\mathbf{R}^d} |K(\mathbf{x},\mathbf{y})|\, d\mathbf{y} < C_2 \ \ (\text{a.e. } \mathbf{x}).$$

Theorem. *If kernel K of the integral operator satisfies*

$$C_1 := \int_{\mathbf{R}^d} \|K(\cdot, \cdot - \mathbf{y})\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\, d\mathbf{y} < \infty, \qquad C_2 := \int_{\mathbf{R}^d} \|K(\cdot - \mathbf{y}, \cdot)\|_{\mathrm{L}^\infty(\mathbf{R}^d)}\, d\mathbf{y} < \infty$$

then it is bounded on $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d),\ \mathbf{p} \in \langle 1, \infty\rangle^d$. ∎

Compactness

We consider

$$\mathrm{H}^{s,\mathbf{p}}(\mathbf{R}^d) = \left\{ u \in \mathcal{S}' : \mathcal{F}^{-1}((1+4\pi^2|\boldsymbol{\xi}|^2)^{\frac{s}{2}}\hat{u}) \in \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \right\}.$$

For two Banach spaces $A_0, A_1 \leqslant X$, we can define a space $(A_0, A_1)_{[\theta]}$ for $\theta \in [0,1]$ by complex interpolation.
First define a vector space $\mathcal{F}(A_0, A_1)$ consisting of all functions of complex variable with values in $A_0 + A_1$, which are bounded and continuous on the closed strip

$$S = \{z \in \mathbf{C} : 0 \leqslant \operatorname{Re} z \leqslant 1\},$$

and analytic on the open strip.

Moreover, the functions $t \mapsto f(j+it)$ are continuous from $\mathbf{R}$ into A_j, and tend to zero as $|t| \longrightarrow \infty$. The norm is

$$\|f\|_{\mathcal{F}} = \max\left\{ \sup_{t\in\mathbf{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbf{R}} \|f(1+it)\|_{A_1} \right\}.$$

Then we define

$$(A_0, A_1)_{[\theta]} = \left\{ a \in A_0 + A_1 \ : \ a = f(\theta) \text{ for some } f \in \mathcal{F}(A_0, A_1) \right\},$$

with the norm

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathcal{F}} \ : \ f(\theta) = a, \ f \in \mathcal{F}(A_0, A_1)\}.$$

Main theorem

Theorem. *Let* $s_0, s_1 \in \mathbf{R}$, $0 < \theta < 1$, *and* $s = (1-\theta)s_0 + \theta s_1$. *Then*

$$\left(\mathrm{H}^{s_0,\mathbf{p}_0}(\mathbf{R}^d), \mathrm{H}^{s_1,\mathbf{p}_1}(\mathbf{R}^d)\right)_{[\theta]} = \mathrm{H}^{s,\mathbf{p}}(\mathbf{R}^d) \ ,$$

for any $\mathbf{1} < \mathbf{p}_0, \mathbf{p}_1 < \boldsymbol{\infty}$, *where* $1/\mathbf{p} = (1-\theta)/\mathbf{p}_0 + \theta/\mathbf{p}_1$. ∎

This, in turn, leads to a form of the Rellich-Kondrašov theorem for mixed-norm spaces

Theorem. *Let* $s_0, s_1 \in \mathbf{R}$, $0 < \theta < 1$, *and* $s = (1-\theta)s_0 + \theta s_1$. *Then*

$$\left(\mathrm{H}^{s_0,\mathbf{p}_0}(\mathbf{R}^d), \mathrm{H}^{s_1,\mathbf{p}_1}(\mathbf{R}^d)\right)_{[\theta]} = \mathrm{H}^{s,\mathbf{p}}(\mathbf{R}^d) \ ,$$

for any $\mathbf{1} < \mathbf{p}_0, \mathbf{p}_1 < \infty$, *where* $1/\mathbf{p} = (1-\theta)/\mathbf{p}_0 + \theta/\mathbf{p}_1$. ∎

Hörmander-Mihlin's theorem for mixed-norm spaces

Theorem. *Let* $m \in \mathrm{L}^\infty(\mathbf{R}^d \setminus \{0\})$ *for some* $A > 0$ *and any* $|\boldsymbol{\alpha}| \leqslant [\frac{d}{2}] + 1$
(a) either Mihlin's condition $|\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})| \leqslant A|\boldsymbol{\xi}|^{-|\boldsymbol{\alpha}|}$ *or*
(b) Hörmander's condition

$$\sup_{R>0} R^{-d+2|\boldsymbol{\alpha}|} \int_{R<|\boldsymbol{\xi}|<2R} |\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leqslant A^2 < \infty \, .$$

Then m *lies in* $\mathcal{M}_{\mathbf{p}}$, *for any* $\mathbf{p} \in \langle 1, \infty \rangle^d$, *and we have the estimate*

$$\begin{aligned} \|m\|_{\mathcal{M}_{\mathbf{p}}} &\leqslant \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max\{p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}}\}(A + \|m\|_{\mathrm{L}^\infty}) \\ &\leqslant c' \prod_{j=0}^{d-1} \max\{p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}}\}(A + \|m\|_{\mathrm{L}^\infty}) \, , \end{aligned}$$

where c *and* c' *are constants that depend only on* d. ∎

[N.A. & I. Ivec (2016)]

First commutation lemma on mixed-norm Lebesgue spaces

Lemma. *Let* (v_n) *be bounded both in* $\mathrm{L}^2(\mathbf{R}^d)$ *and in* $\mathrm{L}^{\mathbf{r}}(\mathbf{R}^d)$*, for some* $\mathbf{r} \in [2, \infty]^d$*, and such that* $v_n \rightharpoonup 0$ *in* $\mathcal{D}'$*. Then* (Cv_n)*, where the commutator is defined by* $C := \mathcal{A}_\psi M_\varphi - M_\varphi \mathcal{A}_\psi$*, strongly converges to zero in* $\mathrm{L}^{\mathbf{q}}(\mathbf{R}^d)$*, for any* $\mathbf{q} \in [2, \infty\rangle^d$ *such that there exists* $\lambda \in \langle 0, 1\rangle$ *for which it holds*

$$\frac{1}{q_i} = \frac{\lambda}{2} + \frac{1-\lambda}{r_i}, \qquad i \in 1..d\,.$$

∎

H-distributions on mixed-norm Lebesgue spaces

Theorem. *Let* $\kappa = [d/2]+1$ *and* $\mathbf{p} \in \langle 1, \infty\rangle^d$. *If* $u_n \longrightarrow 0$ *weakly in* $\mathrm{L}^{\mathbf{p}}_{\rm loc}(\mathbf{R}^d)$, $v_n \xrightarrow{*} v$ *in* $\mathrm{L}^{\mathbf{q}}_{\rm loc}(\mathbf{R}^d)$, *for some* $\mathbf{q} \in [2,\infty]^d$ *such that* $\mathbf{q} > \mathbf{p}'$, *then there exist subsequences* $(u_{n'})$ *and* $(v_{n'})$ *and a complex distribution* $\mu \in \mathcal{D}'(\mathbf{R}^d \times S^{d-1})$, *such that for* $\phi_1, \phi_2 \in \mathrm{C}^\infty_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}^\kappa(S^{d-1})$ *one has*

$$\begin{aligned}\lim_{n'} {}_{\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)}\Big\langle \mathcal{A}_\psi(\phi_1 u_{n'}), \phi_2 v_{n'} \Big\rangle_{\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)} &= \lim_{n'} {}_{\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)}\Big\langle \phi_1 u_{n'}, \mathcal{A}_{\overline{\psi}}(\phi_2 v_{n'}) \Big\rangle_{\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)} \\ &= \langle \mu, \overline{\phi}_1 \phi_2 \boxtimes \overline{\psi}\rangle \;,\end{aligned}$$

where $\mathcal{A}_\psi : \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \longrightarrow \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ *is the Fourier multiplier operator.* ■

Back to consequences of Schwartz theorem

μ is the *H-distribution* corresponding to (a subsequence of) (u_n) and (v_n).

If (u_n), (v_n) are defined on $\Omega \subseteq \mathbf{R}^d$, extension by zero to $\mathbf{R}^d$ preserves the convergence, and we can apply the Theorem. μ is supported on $\mathrm{Cl}\,\Omega \times \mathrm{S}^{d-1}$.

We distinguish $u_n \in \mathrm{L}^p(\mathbf{R}^d)$ and $v_n \in \mathrm{L}^q(\mathbf{R}^d)$. For $p \geqslant 2$, $p' \leqslant 2$ and we can take $q \geqslant 2$; this covers the L^2 case (including $u_n = v_n$).
The assumptions imply $u_n, v_n \longrightarrow 0$ in $\mathrm{L}^2_{\rm loc}(\mathbf{R}^d)$, resulting in a distribution μ of order zero (an unbounded Radon measure, not a general distribution).
The novelty in Theorem is for $p < 2$.

For vector-valued $\mathsf{u}_n \in \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^k)$ and $\mathsf{v}_n \in \mathrm{L}^q(\mathbf{R}^d; \mathbf{C}^l)$, the result is a *matrix valued distribution* $\boldsymbol{\mu} = [\mu^{ij}]$, $i \in 1..k$ and $j \in 1..l$.

The H-distribution would correspond to a non-diagonal block for an H-measure

The proof is based on First commutation lemma

If $q < r$, we can apply the classical interpolation inequality:

$$\|Cv_n\|_q \leqslant \|Cv_n\|_2^{\alpha} \|Cv_n\|_r^{1-\alpha} \ ,$$

for $\alpha \in \langle 0, 1\rangle$ such that $1/q = \alpha/2 + (1-\alpha)/r$. As C is compact on $\mathrm{L}^2(\mathbf{R}^d)$ by Tartar's First commutation lemma, while it is bounded on $\mathrm{L}^r(\mathbf{R}^d)$, we get the claim.

For the most interesting case, where $q = r$, we need a better result: the Krasnosel'skij theorem (a variant of Riesz-Thorin theorem).

In fact, the commutator C is compact on all $\mathrm{L}^p(\mathbf{R}^d)$, $p \in \langle 1, \infty\rangle$.
For that we need an extension of the Krasnosel'skij's result to unbounded domains [N.A., M. Mišur, D. Mitrović (2018)]

Lemma. *Assume that linear operator* A *is compact on* $\mathrm{L}^2(\mathbf{R}^d)$ *and bounded on* $\mathrm{L}^r(\mathbf{R}^d)$*, for some* $r \in \langle 1, \infty\rangle \setminus \{2\}$*. Then* A *is also compact on any* $\mathrm{L}^p(\Omega)$*, where* $1/p = \theta/2 + (1-\theta)/r$*, for a* $\theta \in \langle 0, 1\rangle$*.* ∎