Parabolic H-measures and applications in continuum mechanics

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Why measures in partial differential equations?

Consider a pde like:

$$u_t - \kappa u_{xx} = f \quad \text{on } \mathbf{R}^+ \times [0, L] \, .$$

We seek a function u taking prescribed values at the boundary of the above region. Nothing to do with any measures.

The same goes with the wave equation, or $-\triangle u = f$.

Suppose we want to solve (possibly nonlinear) equation: $\mathcal{A}[u] = f$.
Here, $\mathcal{A}$ is some complicated partial differential operator, and the equation contains some additional conditions (boundary and/or initial).

We might try the following procedure:
Approximate $\mathcal{A}$ by a sequence $\mathcal{A}_n$ of operators we know how to solve, and also f by a sequence f_n of nicer functions, if needed.

Then solve each of the problems: $\mathcal{A}_n[u_n] = f_n$, obtaining the solutions u_n.

It is only natural to expect that the limit $u := \lim u_n$ will be a solution of the original problem.

Of course, this is only a rough idea — in each particular case we have to be more precise. In particular with the definition of various limits taken.

An example: calculus of variations

Here we consider $\mathcal{A}$ as the *derivative* of an appropriate energy functional $\mathcal{I}$, and the solution u is characterised to be the minimiser (or maximiser) of $\mathcal{I}$.

$$\mathcal{I}(u) := \int_0^1 \left(\frac{u^2}{2} + \left(1 - |u'|^2\right)^2 \right) dt \longrightarrow \text{min} ,$$

taking $u \in \mathrm{W}^{1,4}([0,1])$, with $u(0) = u(1) = 0$. Clearly, $\mathcal{I}(u) \geqslant 0$.

However, $\mathcal{I}(u) = 0$ would imply $u = 0$ and $u' = \pm 1$, which is not possible.

As an approximating solution we can take *a minimising sequence* for $\mathcal{I}$, i.e. a sequence such that

$$\lim \mathcal{I}(u_n) = \inf \mathcal{I} = 0 .$$

Such a solution cannot be a function, but only a measure.

Typical examples of spaces in duality

If H is a Hilbert space, any continuous (bounded) linear functional l on it

$$l : H \longrightarrow \mathbf{C}, \qquad \|l\| = \sup_{u \in H} \frac{|l(u)|}{\|u\|} < \infty$$

can be identified by a $v \in H$, such that

$$l(u) = \langle\, v \mid u \,\rangle \;.$$

For $H = \mathrm{L}^2$, we take $\langle\, v \mid u \,\rangle = \int \overline{v} u$.

If B is a Banach space, we have its dual B'. For $B = \mathrm{L}^p$, we also have a representation by $v \in \mathrm{L}^{p'}$, where $1/p + 1/p' = 1$, for $1 \leqslant p < \infty$:

$$l(u) = \langle v, u \rangle = \int \overline{v} u \;.$$

General theory gives us nice properties for the duals of separable Banach spaces: the sequences describe the weak topology of bounded sets, where

$$v_n \rightharpoonup v \iff (\forall\, u \in B)\ \langle v_n, u \rangle \longrightarrow \langle v, u \rangle \;.$$

Furthermore, any bounded sequence has a weakly converging subsequence.

What for $p = 1$ and $p = \infty$?

Actually, the above topologies are weak $*$ topologies, as they are defined on duals by the original spaces. As for $1 < p < \infty$ the spaces are reflexive, they are also weak topologies.

For $p = \infty$, we just have to avoid calling it weak topology, and use the name weak $*$.

The true problem is with $p = 1$. For bounded sets (i.e. on spaces of finite measure), there is a complicated criterion, which is difficult to use.
In practice, we often use the fact that bounded Radon measures $\mathcal{M}_b = (\mathrm{C}_0)'$ behave in a similar way as L^∞, and enclose L^1 in that space.
For $f \in \mathrm{L}^1$ we take $f d\mathbf{x}$ as a measure, where $d\mathbf{x}$ denotes the Lebesgue measure (volume).

When pdes are solved by variational techniques, L^1 terms usually appear under the integral sign. And we have to pass to the limit.

What are H-measures?

Mathematical objects introduced by:

- Luc Tartar, motivated by intended applications in homogenisation (H), and
- Patrick Gérard, whose motivation were certain problems in kinetic theory (and who called these objects *microlocal defect measures*).

Start from $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$, and take the Fourier transform:

$$\widehat{\varphi u_n}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} (\varphi u_n)(\mathbf{x}) d\mathbf{x}$$

As φu_n is supported on a fixed compact set K, so $|\widehat{\varphi u_n}(\boldsymbol{\xi})| \leqslant C$.
Furthermore, $u_n \rightharpoonup 0$, and from the definition $\widehat{\varphi u_n}(\boldsymbol{\xi}) \longrightarrow 0$ pointwise.

By the Lebesgue dominated convergence theorem applied on bounded sets

$$\widehat{\varphi u_n} \longrightarrow 0 \text{ strong, i.e. strongly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d) \; .$$

On the other hand, by the Plancherel theorem: $\|\widehat{\varphi u_n}\|_{\mathrm{L}^2(\mathbf{R}^d)} = \|\varphi u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}$.

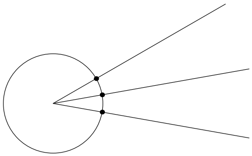
If $\varphi u_n \not\rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, then $\widehat{\varphi u_n} \not\rightharpoonup 0$; some information must go to infinity.

Limit is a measure

How does it go to infinity in various directions? Take $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, and consider:

$$\lim_n \int_{\mathbf{R}^d} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)|\widehat{\varphi u_n}|^2 d\boldsymbol{\xi} = \int_{\mathrm{S}^{d-1}} \psi(\boldsymbol{\xi}) d\nu_\varphi(\boldsymbol{\xi}) \; .$$

The limit is a linear functional in ψ, thus an integral over the sphere of some nonnegative Radon measure (a bounded sequence of Radon measures has an accumulation point), which depends on φ. How does it depend on φ?



Theorem. (u^n) *a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, $\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there is a subsequence* $(\mathsf{u}^{n'})$ *and* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}^{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}^{n'}\Big) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1\bar{\varphi}_2\psi\rangle$$

$$= \int_{\mathbf{R}^d\times\mathrm{S}^{d-1}} \varphi_1(\mathbf{x})\bar{\varphi}_2(\mathbf{x})\psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x},\boldsymbol{\xi}) \; .$$

Why a parabolic variant?

Parabolic pde-s are:

- well studied, and we have good theory for them
- in some cases we even have explicit solutions (by formulae)

$1 : 2$ is certainly a good ratio to start with

Besides the immediate applications (which motivated this research), related to the properties of parabolic equations, applications are possible to other equations and problems involving the scaling $1 : 2$.
Naturally, after understanding this ratio $1 : 2$, other ratios should be considered as well, as required by intended applications.

Terminology: *classical* as opposed to *parabolic or variant* H-measures.
The sphere we replace by:

$$\sigma^4(\tau, \boldsymbol{\xi}) := (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4 = 1 \ , \text{ or}$$

$$\sigma_1^2(\tau, \boldsymbol{\xi}) := |\tau| + (2\pi|\boldsymbol{\xi}|)^2 = 1 \ .$$

finally we chose the ellipse

$$\rho^2(\tau, \boldsymbol{\xi}) := |\boldsymbol{\xi}/2|^2 + \sqrt{(\boldsymbol{\xi}/2)^4 + \tau^2} = 1 \ .$$

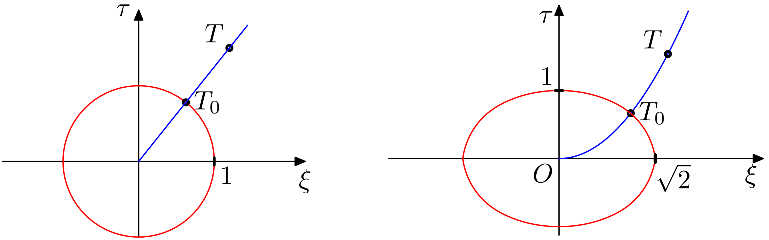
Notation.
For simplicity (2D): $(t, x) = (x^0, x^1) = \mathbf{x}$ and $(\tau, \xi) = (\xi_0, \xi_1) = \boldsymbol{\xi}$.

We use the Fourier transform in both space and time variables.

Rough geometric idea

Take a sequence $u_n \rightharpoonup 0$ in $L^2(\mathbf{R}^2)$, and integrate $|\widehat{\varphi u_n}|^2$ along

rays and project onto S^1 parabolas and project onto P^1



In $\mathbf{R}^2$ we have a compact curve (a surface in higher dimensions):

$$S^1 \ldots \; r^2(\tau,\xi) := \tau^2 + \xi^2 = 1 \qquad P^1 \ldots \; \rho^2(\tau,\xi) := (\xi/2)^2 + \sqrt{(\xi/2)^4 + \tau^2} = 1$$

and projection of $\mathbf{R}^2_* = \mathbf{R}^2 \setminus \{0\}$ onto the curve (surface):

$$p(\tau,\xi) := \left(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\right) \qquad \pi(\tau,\xi) := \left(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\right)$$

Analytic picture

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier P_a, for $a \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{P_a u} = a\hat{u}$.
The norm is again equal to $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Delicate part: a is given only on S^1 or P^1.
We extend it by the projections, p or π: if α is a function defined on a compact surface, we take $a := \alpha \circ p$ or $a := \alpha \circ \pi$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau, \xi)}, \frac{\xi}{r(\tau, \xi)}\Big) \qquad a(\tau, \xi) := \alpha\Big(\frac{\tau}{\rho^2(\tau, \xi)}, \frac{\xi}{\rho(\tau, \xi)}\Big)$$

The precise scaling is contained in the projections, not the surface.

Now we can state the main theorem.

Existence of H-measures

Theorem. If $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, then there exists its subsequence and a complex matrix Radon measure $\boldsymbol{\mu}$ on

$$\mathbf{R}^d \times S^{d-1} \qquad\qquad \mathbf{R}^d \times P^{d-1}$$

such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_0(\mathbf{R}^d)$ and

$$\psi \in \mathrm{C}(S^{d-1}) \qquad\qquad \psi \in \mathrm{C}(P^{d-1})$$

one has

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} (\psi \circ p\pi)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times S^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi}) \qquad = \int_{\mathbf{R}^d \times P^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi}) \,.$$

Oscillation (classical H-measures)

$$u_n(\mathbf{x}) := v(n\mathbf{x}) \rightharpoonup 0$$

$v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$ periodic function (with the unit period in each of variables), with the zero mean value.

The associated H-measure

$$\mu(\mathbf{x}, \boldsymbol{\xi}) = \sum_{\mathsf{k}\in\mathbf{Z}^d\setminus\{0\}} |v_{\mathsf{k}}|^2 \lambda(\mathbf{x})\, \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}(\boldsymbol{\xi}),$$

v_{k} Fourier coefficients of v ($v(\mathbf{x}) = \sum_{\mathsf{k}\in\mathbf{Z}^d} v_{\mathsf{k}} e^{2\pi i \mathsf{k}\cdot\mathbf{x}}$).

Dual variable *preserves* information on the direction of propagation (of oscillation).

Here λ denotes the Lebesgue mesure (volume, area or length).

Oscillation (parabolic H-measures)

Let $v \in \mathrm{L}^2(\mathbf{R}^{1+d})$ be a periodic function

$$v(t, \mathbf{x}) = \sum_{(\omega, \mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega, \mathsf{k}} \, e^{2\pi i(\omega t + \mathsf{k} \cdot \mathbf{x})} \; ,$$

where $\hat{v}_{\omega, \mathsf{k}}$ denotes Fourier coefficients. Further, assume that v has mean value zero, i.e. $\hat{v}_{0,0} = 0$.

For $\alpha, \beta \in \mathbf{R}^+$, we have a sequence of periodic functions with period tending to zero:

$$u_n(t, \mathbf{x}) := v(n^\alpha t, n^\beta \mathbf{x}) = \sum_{(\omega, \mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega, \mathsf{k}} \, e^{2\pi i(n^\alpha \omega t + n^\beta \mathsf{k} \cdot \mathbf{x})} \; .$$

Their Fourier transforms are:

$$\hat{u}_n(\tau, \boldsymbol{\xi}) = \sum_{(\omega, \mathsf{k}) \in \mathbf{Z}^{1+d}} \hat{v}_{\omega, \mathsf{k}} \, \delta_{n^\alpha \omega}(\tau) \delta_{n^\beta \mathsf{k}}(\boldsymbol{\xi}) \; .$$

Oscillation (cont.)

(u_n) is a pure sequence, and the corresponding parabolic H-measure $\mu(t, \mathbf{x}, \tau, \boldsymbol{\xi})$ is

$$
\lambda(t,\mathbf{x}) \begin{cases} \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \omega\neq 0}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}) + \sum_{\mathsf{k}\in\mathbf{Z}^d} |\hat{v}_{0,\mathsf{k}}|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}), & \alpha > 2\beta \\ \displaystyle\sum_{\substack{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}\\ \mathsf{k}\neq 0}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{(0,\frac{\mathsf{k}}{|\mathsf{k}|})}(\tau,\boldsymbol{\xi}) + \sum_{\omega\in\mathbf{Z}} |\hat{v}_{\omega,0}|^2 \delta_{(\frac{\omega}{|\omega|},0)}(\tau,\boldsymbol{\xi}), & \alpha < 2\beta \\ \displaystyle\sum_{(\omega,\mathsf{k})\in\mathbf{Z}^{1+d}} |\hat{v}_{\omega,\mathsf{k}}|^2 \delta_{\left(\frac{\omega}{\rho^2(\omega,\mathsf{k})},\frac{\mathsf{k}}{\rho(\omega,\mathsf{k})}\right)}(\tau,\boldsymbol{\xi}), & \alpha = 2\beta, \end{cases}
$$

where λ denotes the Lebesgue measure.

Concentration (classical H-measures)

$$u_n(\mathbf{x}) := n^{\frac{d}{2}} v\,(n\mathbf{x})\,, \qquad \Big(v \in \mathrm{L}^2(\mathbf{R}^d)\Big)\,.$$

The associated H-measure is of the form $\delta_0(\mathbf{x})\nu(\boldsymbol{\xi})$, where ν is measure on S^{d-1} with surface density

$$\nu(\boldsymbol{\xi}) = \int_0^\infty |\hat{v}(t\boldsymbol{\xi})|^2 t^{d-1} dt,$$

i.e.

$$\mu(\mathbf{x},\boldsymbol{\xi}) = \int_{\mathbf{R}^d} |\hat{v}(\boldsymbol{\eta})|^2 \delta_{\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}}(\boldsymbol{\xi})\delta_0(\mathbf{x})\,d\boldsymbol{\eta},$$

where $\hat{v}$ denotes the Fourier transformation of v.

Concentration (parabolic H-measures)

For $v \in \mathrm{L}^2(\mathbf{R}^{1+d})$ and $\alpha, \beta \in \mathbf{R}^+$

$$u_n(t, \mathbf{x}) := n^{\alpha+\beta d} v(n^{2\alpha} t, n^{2\beta} \mathbf{x}),$$

is bounded in $\mathrm{L}^2(\mathbf{R}^{1+d})$ with the norm $\|u_n\|_{\mathrm{L}^2(\mathbf{R}^{1+d})} = \|v\|_{\mathrm{L}^2(\mathbf{R}^{1+d})}$ which does not depend on n, and weakly converges to zero.

(u_n) is a pure sequence, with the parabolic H-measure $\langle \boldsymbol{\mu}, \phi \boxtimes \psi \rangle =$

$$\phi(0,0) \begin{cases} \int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi(\frac{\sigma}{|\sigma|}, 0) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}^d} |\hat{v}(0, \boldsymbol{\eta})|^2 \psi(0, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|})\, d\boldsymbol{\eta}, & \alpha > 2\beta \\ \int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi(0, \frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}) d\sigma d\boldsymbol{\eta} + \int_{\mathbf{R}} |\hat{v}(\sigma, 0)|^2 \psi(\frac{\sigma}{|\sigma|}, 0)\, d\sigma, & \alpha < 2\beta \\ \int_{\mathbf{R}^{1+d}} |\hat{v}(\sigma, \boldsymbol{\eta})|^2 \psi\left(\frac{\sigma}{\rho^2(\sigma, \boldsymbol{\eta})}, \frac{\boldsymbol{\eta}}{\rho(\sigma, \boldsymbol{\eta})}\right) d\sigma d\boldsymbol{\eta}, & \alpha = 2\beta. \end{cases}$$

From examples we learn . . .

Actually, any non-negative Radon measure on $\Omega \times P^{d-1}$, of total mass A^2, can be described as a parabolic H-measure of some sequence $u_n \rightharpoonup 0$, with $\|u_n\|_{\mathrm{L}^2} \leqslant A + \varepsilon$.

Both for oscillation and concentration, for $\alpha > 2\beta$ the measure μ is supported in *poles*, while for $\alpha < 2\beta$ on the *equator* of the surface P^d, regardless of the choice of v.

When $\alpha = 2\beta$ the parabolic H-measure can be supported in any point of the surface P^d.

Other research in this direction:
Panov (IHP:AN, 2011): ultraparabolic H-measures
Ivec & Mitrović (CPAA, 2011): general surfaces
Lazar & Mitrović (MathComm, 2011): applications to velocity averaging
Erceg & Ivec (JPsiDO, Filomat, 2017): fractional H-measures
Ivec & Lazar (JPsiDO, 2019): propagation principle

Symmetric systems — localisation principle

$$\sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}) + \mathbf{B}\mathsf{u} = \mathsf{f} \ , \ \mathbf{A}^k \in \mathrm{C}_b(\mathbf{R}^d; \mathrm{M}_{r\times r}) \text{ Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0 \ , \qquad \text{and defines } \boldsymbol{\mu}$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \ .$$

Theorem. (localisation principle) If u^n satisfies:

$$\partial_k\left(\mathbf{A}^k \mathsf{u}^n\right) \longrightarrow 0 \ \text{ in space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ on $\Omega \times S^{d-1}$ one has:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\bar{\boldsymbol{\mu}} = \mathbf{0} \ .$$

Thus, the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{P}$ is a singular matrix.

The localisation principle is behind the applications to the small-amplitude homogenisation, which can be used in optimal design.

It is a generalisation of compactness by compensation to variable coefficients.

Localisation principle for parabolic H-measures

In the parabolic case the details become more involved.

Anisotropic Sobolev spaces ($s \in \mathbf{R}$; $k_p(\tau, \boldsymbol{\xi}) := (1 + \sigma^4(\tau, \boldsymbol{\xi}))^{1/4}$))

$$\mathrm{H}^{\frac{s}{2},s}(\mathbf{R}^{1+d}) := \left\{ u \in \mathcal{S}' : k_p^s \hat{u} \in \mathrm{L}^2(\mathbf{R}^{1+d}) \right\} .$$

Theorem. (localisation principle) Let $\mathsf{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{C}^r)$, uniformly compactly supported in t, satisfy ($s \in \mathbf{N}$)

$$\sqrt{\partial_t}^{\,s}(\mathsf{u}_n \cdot \mathsf{b}) + \sum_{|\boldsymbol{\alpha}|=s} \partial_{\mathbf{x}}^{\boldsymbol{\alpha}}(\mathsf{u}_n \cdot \mathsf{a}_{\boldsymbol{\alpha}}) \longrightarrow 0 \quad \text{in} \quad \mathrm{H}_{\mathrm{loc}}^{-\frac{s}{2},-s}(\mathbf{R}^{1+d}) ,$$

where $\mathsf{b}, \mathsf{a}_{\boldsymbol{\alpha}} \in \mathrm{C}_b(\mathbf{R}^{1+d}; \mathbf{C}^r)$, while $\sqrt{\partial_t}$ is a pseudodifferential operator with polyhomogeneous symbol $\sqrt{2\pi i \tau}$, i.e.

$$\sqrt{\partial}_t u = \overline{\mathcal{F}} \left(\sqrt{2\pi i \tau}\, \hat{u}(\tau) \right) .$$

For parabolic H-measure $\boldsymbol{\mu}$ associated to sequence (u_n) one has

$$\boldsymbol{\mu} \left((\sqrt{2\pi i \tau})^s \bar{\mathsf{b}} + \sum_{|\boldsymbol{\alpha}|=s} (2\pi i \boldsymbol{\xi})^{\boldsymbol{\alpha}}\, \bar{\mathsf{a}}_{\boldsymbol{\alpha}} \right) = 0.$$

How to use such a relation? — the heat equation

$$\begin{cases} \partial_t u_n - \operatorname{div}(\mathbf{A}\nabla u_n) = \operatorname{div} \mathsf{f}_n \\ u_n(0) = \gamma_n \ , \end{cases}$$

$\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^{1+d};\mathbf{R}^d)$, $\gamma_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$

continuous, bounded and positive definite: $\mathbf{A}(t,\mathbf{x})\,\mathsf{v}\cdot\mathsf{v} \geqslant \alpha \mathsf{v}\cdot\mathsf{v}$

Localise in time: take θu_n, for $\theta \in \mathrm{C}^1_c(\mathbf{R}^+)$, ...
Now we can apply the localisation principle (we still denote the localised solutions by u_n).

Furthermore, $\sqrt{\partial_t}\,(u_n) := \left(\sqrt{2\pi i\tau}\,\widehat{u_n}\right)^{\vee} \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+d})$.

The heat equation (cont.)

Take

$$\tilde{\mathsf{v}}_n = (v_n^0, \mathsf{v}_n, \mathsf{f}_n) := (\sqrt{\partial_t} u_n, \nabla u_n, \mathsf{f}_n) \rightharpoonup 0$$

in $\mathrm{L}^2(\mathbf{R}^{1+d}; \mathbf{R}^{1+2d})$, which (on a subsequence) defines H-measure

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \mu_0 & \boldsymbol{\mu}_{01} & \boldsymbol{\mu}_{02} \\ \boldsymbol{\mu}_{10} & \boldsymbol{\mu} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{20} & \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_f \end{bmatrix}.$$

The localisation principle gives us:

$$\begin{aligned} \mu_0 \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu}_{01} \cdot \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_{02} \cdot \boldsymbol{\xi} &= 0 \\ \boldsymbol{\mu}_{10} \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu} \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_{12} \boldsymbol{\xi} &= 0 \\ \boldsymbol{\mu}_{20} \sqrt{2\pi i \tau} - 2\pi i \boldsymbol{\mu}_{21} \mathbf{A}^\top \boldsymbol{\xi} - 2\pi i \boldsymbol{\mu}_f \boldsymbol{\xi} &= 0. \end{aligned}$$

After some calculation (linear algebra) ...

Expression for H-measure — from given data

$$\mathrm{tr}\boldsymbol{\mu} = \frac{(2\pi\boldsymbol{\xi})^2}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi},$$

$$\boldsymbol{\mu} = \frac{(2\pi)^2}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}(\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi})\boldsymbol{\xi}\otimes\boldsymbol{\xi}.$$

$$\mu_0 = \frac{|2\pi\tau|}{\tau^2 + (2\pi\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi})^2}\boldsymbol{\mu}_f\boldsymbol{\xi}\cdot\boldsymbol{\xi}.$$

Thus, from the H-measures for the right hand side term f one can calculate the H-measure of the solution.

However, the oscillation in initial data dies out (the equation is hypoelliptic). Only the right hand side affects the H-measure of solutions.

The situation is different for the Schrödinger equation and for the vibrating plate equation.

Small amplitude homogenisation: setting of the problem

A sequence of parabolic problems

$$(*) \qquad \begin{cases} \partial_t u_n - \mathsf{div}\,(\mathbf{A}^n \nabla u_n) = f \\ \qquad\qquad u_n(0,\cdot) = u_0 \ . \end{cases}$$

where $\mathbf{A}^n$ is a perturbation of $\mathbf{A}_0 \in \mathrm{C}(Q; \mathrm{M}_{d\times d})$, $Q := \langle 0, T\rangle \times \Omega$, which is bounded from below; for small γ function $\mathbf{A}^n$ is analytic in γ:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma \mathbf{B}^n(t,\mathbf{x}) + \gamma^2 \mathbf{C}^n(t,\mathbf{x}) + o(\gamma^2) \ ,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{\ *\ } \mathbf{0}$ in $\mathrm{L}^\infty(Q; \mathrm{M}_{d\times d})$).

Then (after passing to a subsequence if needed)

$$\mathbf{A}^n_\gamma \xrightarrow{\ H\ } \mathbf{A}^\infty_\gamma = \mathbf{A}_0 + \gamma \mathbf{B}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2) \ ;$$

the limit being measurable in $t, \mathbf{x}$, and analytic in γ.

No first-order term on the limit

Theorem. *The effective conductivity matrix* $\mathbf{A}_\gamma^\infty$ *admits the expansion*

$$\mathbf{A}_\gamma^\infty(t,\mathbf{x}) = \mathbf{A}_0(t,\mathbf{x}) + \gamma^2 \mathbf{C}_0(t,\mathbf{x}) + o(\gamma^2)\ .$$

∎

Indeed, take $u \in \mathrm{L}^2([0,T];\mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle;\mathrm{H}^{-1}(\Omega))$, and define $f_\gamma := \partial_t u - \mathsf{div}\,(\mathbf{A}_\gamma^\infty \nabla u)$, and $u_0 := u(0,\cdot) \in \mathrm{L}^2(\Omega)$.

Next, solve $(*)$ with $\mathbf{A}_\gamma^n$, f_γ and u_0, the solution u_γ^n.
Of course, f_γ and u_γ^n analytically depend on γ.

Because of H-convergence, we have the weak convergences in $\mathrm{L}^2(Q)$:

$$(\dagger) \qquad \begin{aligned} \mathsf{E}_\gamma^n &:= \nabla u_\gamma^n \rightharpoonup \nabla u \\ \mathsf{D}_\gamma^n &:= \mathbf{A}_\gamma^n \mathsf{E}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u\ . \end{aligned}$$

Expansions in Taylor series (similarly for f_γ and u_γ^n):

$$\begin{aligned} \mathsf{E}_\gamma^n &= \mathsf{E}_0^n + \gamma \mathsf{E}_1^n + \gamma^2 \mathsf{E}_2^n + o(\gamma^2) \\ \mathsf{D}_\gamma^n &= \mathsf{D}_0^n + \gamma \mathsf{D}_1^n + \gamma^2 \mathsf{D}_2^n + o(\gamma^2)\ . \end{aligned}$$

No first-order term on the limit (cont.)

Inserting (†) and equating the terms with equal powers of γ:

$$\mathsf{E}_0^n = \nabla u \,, \qquad \mathsf{D}_0^n = \mathbf{A}_0 \nabla u$$

$$\mathsf{D}_1^n = \mathbf{A}_0 \mathsf{E}_1^n + \mathbf{B}^n \nabla u \rightharpoonup 0 \quad \text{in } \mathrm{L}^2(Q) \,.$$

Also, D_1^n converges to $\mathbf{B}_0 \nabla u$ (the term in expansion with γ^1)

$$\mathsf{D}_\gamma^n \rightharpoonup \mathbf{A}_\gamma^\infty \nabla u = \mathbf{A}_0 \nabla u + \gamma \mathbf{B}_0 \nabla u + \gamma^2 \mathbf{C}_0 \nabla u + o(\gamma^2) \,.$$

Thus $\mathbf{B}_0 \nabla u = 0$, and as $u \in \mathrm{L}^2([0,T]; \mathrm{H}_0^1(\Omega)) \cap \mathrm{H}^1(\langle 0,T\rangle; \mathrm{H}^{-1}(\Omega))$ was arbitrary, we conclude that $\mathbf{B}_0 = \mathbf{0}$.
For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0 \mathsf{E}_2^n + \mathbf{B}^n \mathsf{E}_1^n + \mathbf{C}^n \nabla u \rightharpoonup \lim \mathbf{B}^n \mathsf{E}_1^n = \mathbf{C}_0 \nabla u \,,$$

and this is the limit we still have to compute.

Periodic homogenisation — an example

In the periodic case the explicit formulae for the homogenisation limit are known [BLP].
Together with Fourier analysis:
leading terms in expansion for the small amplitude homogenisation limit.

Periodic functions—functions defined on $T := S^1 = \mathbf{R}/\mathbf{Z}$, $Y := \mathbf{R}^d/\mathbf{Z}^d$ and $Z := \mathbf{R}^{1+d}/\mathbf{Z}^{1+d}$
We implicitly assume projections of $\mathbf{x} \mapsto \mathbf{y} \in Y$, etc.
For given $\rho \in \langle 0, \infty \rangle$ we define the sequence $\mathbf{A}_n$ by

$$\mathbf{A}_n(t, \mathbf{x}) = \mathbf{A}(n^\rho t, n\mathbf{x}) .$$

Then $\mathbf{A}_n$ H-converges to a constant $\mathbf{A}_\infty$ defined by

$$\mathbf{A}_\infty \mathsf{h} = \int_Z \mathbf{A}(\tau, \mathbf{y})(\mathsf{h} + \nabla w(\tau, \mathbf{y}))\, d\tau d\mathbf{y} .$$

For given h, w is a solution of some BVP, depending on ρ.

Three different cases depending on ρ

$\rho \in \langle 0, 2\rangle$: $w(\tau, \cdot)$ is the unique solution of

$$-\text{div}\,(\mathbf{A}(\tau,\cdot)(\mathsf{h} + \nabla w(\tau,\cdot))) = 0$$
$$w(\tau,\cdot) \in \mathrm{H}^1(Y)\,,\ \int_Y w(\tau,\mathbf{y})\,d\mathbf{y} = 0\,,$$

$\rho = 2$: w is defined by

$$\partial_t w - \text{div}\,(\mathbf{A}(\mathsf{h} + \nabla w)) = 0$$
$$w \in \mathrm{L}^2(T;\mathrm{H}^1(Y))\,,\ \partial_t w \in \mathrm{L}^2(T;\mathrm{H}^{-1}(Y))\,,\ \int_Z w\,d\tau d\mathbf{y} = 0\,.$$

$\rho \in \langle 2, \infty\rangle$: define $\widetilde{\mathbf{A}}(y) = \int_0^1 \mathbf{A}(\tau,\mathbf{y})\,d\tau$ and w as the solution of

$$-\text{div}\,(\widetilde{\mathbf{A}}(\mathsf{h} + \nabla w)) = 0$$
$$w \in \mathrm{H}^1(Y)\,,\ \int_Y w\,d\mathbf{y} = 0\,.$$

Periodic small-amplitude homogenisation

A sequence of small perturbations of a constant coercive matrix $\mathbf{A}_0 \in \mathrm{M}_{d\times d}$:

$$\mathbf{A}^n_\gamma(t,\mathbf{x}) = \mathbf{A}_0 + \gamma\mathbf{B}^n(t,\mathbf{x})\,,$$

where $\mathbf{B}^n(t,\mathbf{x}) = \mathbf{B}(n^\rho t, n\mathbf{x})$, $\mathbf{B}$ is Z-periodic L^∞ matrix function satisfying $\int_Z \mathbf{B}\,d\tau d\mathbf{y} = 0$.

For γ small enough, (eventually passing to a subsequence) we have H-convergence to a limit depending analytically on γ:

$$\mathbf{A}^n_\gamma \xrightarrow{H} \mathbf{A}^\infty_\gamma = \mathbf{A}_0 + \gamma\mathbf{B}_0 + \gamma^2\mathbf{C}_0 + o(\gamma^2)$$

and a formula for $\mathbf{A}^\infty_\gamma$:

$$\begin{aligned}\mathbf{A}^\infty_\gamma \mathsf{h} &= \int_Z (\mathbf{A}_0 + \gamma\mathbf{B}))(\mathsf{h} + \nabla w_\gamma)\,d\tau d\mathbf{y}\\ &= \mathbf{A}_0\mathsf{h} + \int_Z \mathbf{A}_0\nabla w_\gamma + \gamma\int_Z \mathbf{B}\mathsf{h} + \gamma\int_Z \mathbf{B}\nabla w_\gamma = \mathbf{A}_0\mathsf{h} + \gamma\int_Z \mathbf{B}\nabla w_\gamma\,.\end{aligned}$$

In the last equality the second term equals zero by Gauss' theorem, as w_γ is a periodic function. Similarly for the third term.

Since w_γ is a solution of some (initial–)boundary value problem, depending on ρ, it also depends analytically on γ:

$$w_\gamma = w_0 + \gamma w_1 + o(\gamma)\,.$$

The first order term vanishes, as $\mathbf{A}_0$ is constant.

$$\mathbf{A}_\gamma^\infty \mathsf{h} = \mathbf{A}_0 \mathsf{h} + \gamma^2 \int_Z \mathbf{B}\nabla w_1 + o(\gamma^2)\,,$$

so we conclude that $\mathbf{B}_0 = \mathbf{0}$ and $\mathbf{C}_0 \mathsf{h} = \int_Z \mathbf{B}\nabla w_1$.
From this formula, using the Fourier series, one can calculate the second-term approximation $\mathbf{C}_0$. Off course, we must treat separately each one of the above three cases for ρ.

The case $\rho \in \langle 0, 2 \rangle$ on the limit

Fix $\tau \in [0,1]$; the BVP with coefficient $\mathbf{A}_0 + \gamma\mathbf{B}$ instead of $\mathbf{A}$ and the above expression for w, we see that w_1 solves

$$(\ddagger) \quad -\mathsf{div}\,(\mathbf{A}_0 \nabla w_1(\tau,\cdot)) = \mathsf{div}\,(\mathbf{B}\mathsf{h})\,,\ w_1(\tau,\cdot) \in \mathrm{H}^1(Y)\,,\ \int_Y w_1(\tau,\mathbf{y})\,d\mathbf{y} = 0$$

Expanding w_1 in the Fourier series gives us ($J = \mathbf{Z} \times (\mathbf{Z}^d \setminus \{\mathbf{0}\})$)

$$w_1 = \sum_{(l,\mathsf{k}) \in J} a_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}\,,$$

because of $\int_Y w_1(\tau,\mathbf{y})\,d\mathbf{y} = 0$.
Straightforward calculation gives us

$$\nabla w_1 = \sum_J 2\pi i \mathsf{k}\, a_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}\,,$$

$$\mathsf{div}\,\mathbf{A}_0 \nabla w_1 = \sum_J (2\pi i)^2 \mathbf{A}_0 \mathsf{k}\cdot\mathsf{k}\, a_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}\,.$$

The case $\rho \in \langle 0, 2\rangle$ on the limit (cont.)

For $\mathbf{B}$ denote $I := \mathbf{Z}^{d+1} \setminus \{\mathbf{0}\}$

$$\mathbf{B} = \sum_{I} \mathbf{B}_{l\mathsf{k}} e^{2\pi i(l\tau + \mathsf{k}\cdot\mathbf{y})} ,$$

$$\operatorname{div} \mathbf{B}\mathsf{h} = \sum_{I} 2\pi i\, \mathbf{B}_{l\mathsf{k}}\mathsf{h} \cdot \mathsf{k}\, e^{2\pi i(l\tau + \mathsf{k}\cdot\mathbf{y})} .$$

(‡) leads to a relation among corresponding Fourier coefficients

$$2\pi i \mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}\, a_{l\mathsf{k}} = -\mathbf{B}_{l\mathsf{k}}\mathsf{h} \cdot \mathsf{k} , \quad (l, \mathsf{k}) \in \mathbf{Z}^{d+1} ,$$

$$\text{i.e.} \quad a_{l\mathsf{k}} = \begin{cases} -\dfrac{\mathbf{B}_{l\mathsf{k}}\mathsf{h} \cdot \mathsf{k}}{2\pi i \mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} , & (l, \mathsf{k}) \in J \\ 0 , & \text{otherwise} . \end{cases}$$

Finally, we obtain

$$\begin{aligned} \mathbf{C}_0\mathsf{h} &= \int_Z \mathbf{B}\nabla w_1 \, d\tau d\mathbf{y} \\ &= \int_Z \Big(\sum_I \mathbf{B}_{l\mathsf{k}} e^{2\pi i(l\tau + \mathsf{k}\cdot\mathbf{y})} \Big) \Big(\sum_J (2\pi i \mathsf{k}') a_{l'\mathsf{k}'} e^{2\pi i(l'\tau + \mathsf{k}'\cdot\mathbf{y})} \Big) \, d\tau d\mathbf{y} \end{aligned}$$

The case $\rho \in \langle 0, 2 \rangle$ on the limit (cont.)

Due to orthogonality, for the non-vanishing terms in the above product of two series we have $l' = -l$ and $\mathsf{k}' = -\mathsf{k}$. Therefore,

$$\begin{aligned}\mathbf{C}_0 \mathsf{h} &= -2\pi i \sum_J \mathbf{B}_{l\mathsf{k}} \mathsf{k} a_{-l,-\mathsf{k}} \\ &= -\sum_J \mathbf{B}_{l\mathsf{k}} \mathsf{k} \frac{\mathbf{B}_{-l,-\mathsf{k}} \mathsf{h} \cdot \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}} \mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} \mathsf{h}\,,\end{aligned}$$

where the last equality holds since $\mathbf{B}$ is a real matrix function i.e. $\overline{\mathbf{B}_{l\mathsf{k}}} = \mathbf{B}_{-l,-\mathsf{k}}$. We conclude

$$\mathbf{C}_0 = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}} \mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}}\,.$$

The case $\rho = 2$ on the limit

The calculation is similar to the first case. The only difference appears in the equation for $w_1 = \sum_{(l,\mathsf{k}) \in I} a_{l\mathsf{k}} e^{2\pi i (l\tau + \mathsf{k}\cdot\mathbf{y})}$:

$$\partial_\tau w_1 - \mathsf{div}\left(\mathbf{A}_0 \nabla w_1(\tau, \cdot)\right) = \mathsf{div}\left(\mathbf{B}\mathsf{h}\right),$$

implying the following relation for the Fourier coefficients

$$(l - 2\pi i \mathbf{A}_0 \mathsf{k} \cdot \mathsf{k} a_{l\mathsf{k}}) = \mathbf{B}_{l\mathsf{k}} \mathsf{h} \cdot \mathsf{k}\,, \quad (l, \mathsf{k}) \in I\,,$$

and the formula for the second order approximation of the H-limit:

$$\mathbf{C}_0 = -\sum_J \frac{\mathbf{B}_{l\mathsf{k}}\mathsf{k} \otimes \mathbf{B}_{l\mathsf{k}}\mathsf{k}}{\frac{l}{2\pi i} + \mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}}\,.$$

The case $\rho \in \langle 2, \infty \rangle$ on the limit

In this case w_1 does not depend on τ. Introducing

$$\widetilde{\mathbf{B}}(\mathbf{y}) := \int_0^1 \mathbf{B}(\tau, \mathbf{y}) \, d\tau$$

this case actually has the same behaviour as the one in elliptic setting, giving the formula

$$\mathbf{C}_0 = - \sum_{\mathbf{Z}^d \setminus \{\mathbf{0}\}} \frac{\widetilde{\mathbf{B}}_{\mathsf{k}} \mathsf{k} \otimes \widetilde{\mathbf{B}}_{\mathsf{k}} \mathsf{k}}{\mathbf{A}_0 \mathsf{k} \cdot \mathsf{k}} \, .$$

Parabolic small-amplitude homogenisation—general case

Let us continue what we were doing before ...
For the quadratic term we have:

$$\mathsf{D}_2^n = \mathbf{A}_0\mathsf{E}_2^n + \mathbf{B}^n\mathsf{E}_1^n + \mathbf{C}^n\nabla u \longrightarrow \lim \mathbf{B}^n\mathsf{E}_1^n = \mathbf{C}_0\nabla u \; ,$$

and this is the limit we shall express using only the parabolic variant H-measure $\boldsymbol{\mu}$.

u_1^n satisfies the equation $(*)$ with coefficients $\mathbf{A}_0$, $\operatorname{div}(\mathbf{B}^n\nabla u)$ on the right hand side and the homogeneous innitial condition.

By applying the Fourier transform (as if the equation were valid in the whole space), and multiplying by $2\pi i\boldsymbol{\xi}$, for $(\tau,\boldsymbol{\xi}) \neq (0,0)$ we get

$$\widehat{\nabla u_1^n}(\tau,\boldsymbol{\xi}) = -\frac{(2\pi)^2\,(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\,(\widehat{\mathbf{B}^n\nabla u})(\tau,\boldsymbol{\xi})}{2\pi i\tau + (2\pi)^2\mathbf{A}_0\boldsymbol{\xi}\cdot\boldsymbol{\xi}}.$$

(the precise argument involves localisation principle and some calculations ...)

Expression for the quadratic correction

As $(\boldsymbol{\xi} \otimes \boldsymbol{\xi})/(2\pi i\tau + (2\pi)^2 \mathbf{A}_0\boldsymbol{\xi} \cdot \boldsymbol{\xi})$ is constant along branches of paraboloids $\tau = c\boldsymbol{\xi}^2, c \in \overline{\mathbf{R}}$, we have ($\varphi \in \mathrm{C}_c^\infty(Q)$)

$$\lim_n \Big\langle\, \varphi \mathbf{B}^n \mid \nabla u_1^n \,\Big\rangle = -\lim_n \Big\langle\, \widehat{\varphi \mathbf{B}^n} \mid \frac{(2\pi)^2\,(\boldsymbol{\xi} \otimes \boldsymbol{\xi})\,(\widehat{\mathbf{B}^n \nabla u})}{2\pi i\tau + (2\pi)^2 \mathbf{A}_0\boldsymbol{\xi} \cdot \boldsymbol{\xi}} \,\Big\rangle$$

$$= -\Big\langle \boldsymbol{\mu}, \varphi \frac{(2\pi)^2 \boldsymbol{\xi} \otimes \boldsymbol{\xi} \otimes \nabla u}{-2\pi i\tau + (2\pi)^2 \mathbf{A}_0\boldsymbol{\xi} \cdot \boldsymbol{\xi}} \Big\rangle,$$

where $\boldsymbol{\mu}$ is the parabolic variant H-measure associated to $(\mathbf{B}^n)$, a measure with four indices (the first two of them not being contracted above).

Expression for the quadratic correction (cont.)

By varying function $u \in \mathrm{C}^1(Q)$ (e.g. choosing ∇u constant on $\langle 0, T\rangle \times \omega$, where $\omega \subseteq \Omega$) we get

$$\int_{\langle 0,T\rangle \times \omega} C_0^{ij}(t, \mathbf{x})\phi(t, \mathbf{x}) dt d\mathbf{x} = -\Big\langle \boldsymbol{\mu}^{ij}, \phi \frac{(2\pi)^2 \boldsymbol{\xi} \otimes \boldsymbol{\xi}}{-2\pi i\tau + (2\pi)^2 \mathbf{A}_0 \boldsymbol{\xi} \cdot \boldsymbol{\xi}} \Big\rangle,$$

where $\boldsymbol{\mu}^{ij}$ denotes the matrix measure with components $\left(\boldsymbol{\mu}^{ij}\right)_{kl} = \mu_{iklj}$.

For the periodic example of small-amplitude homogenisation, we get the same results by applying the variant H-measures, as with direct calculations performed above.

Homogenisation of a model based on the Stokes equation: stationary case

(Tartar, 1976 and 1984)
$\Omega \subseteq \mathbf{R}^3$ open set, $\mathsf{u}_n \longrightarrow \mathsf{u}_0$ in $\mathrm{H}^1_{\mathrm{loc}}(\Omega; \mathbf{R}^3)$

$$\begin{cases} -\nu \triangle \mathsf{u}_n + \mathsf{u}_n \times \operatorname{rot}(\mathsf{v}_0 + \lambda \mathsf{v}_n) + \nabla p_n = \mathsf{f}_n \\ \operatorname{div} \mathsf{u}_n = 0 \,. \end{cases}$$

Not a realistic model, but contains the terms: $\mathsf{u} \times \operatorname{rot} \mathsf{A}$
resulting from the Lorentz force $q(\mathsf{u} \times \mathsf{B})$ in electrostatics, or
in fluids $(\nabla \mathsf{u})\mathsf{u} = \mathsf{u} \times \operatorname{rot}(-\mathsf{u}) + \nabla \frac{|\mathsf{u}|^2}{2}$.

Theorem. *There is a subsequence and* $\mathbf{M} \geqslant 0$*, depending on the choice of the subsequence, such that the limit* u_0 *satisfies:*

$$\begin{cases} -\nu \triangle \mathsf{u}_0 + \mathsf{u}_0 \times \operatorname{rot} \mathsf{v}_0 + \lambda^2 \mathbf{M} \mathsf{u}_0 + \nabla p_0 = \mathsf{f}_0 \\ \operatorname{div} \mathsf{u}_0 = 0 \,, \end{cases}$$

and it holds:

$$\nu |\nabla \mathsf{u}_n|^2 \longrightarrow \nu |\nabla \mathsf{u}_0|^2 + \lambda^2 \mathbf{M} \mathsf{u}_0 \cdot \mathsf{u}_0 \qquad \text{in } \mathcal{D}'(\Omega) \,.$$

∎

Explicit formula via H-measures

Can $\mathbf{M}$ be computed directly from $\mathsf{v}_n \rightharpoonup 0$ in $\mathrm{L}^2(\Omega;\mathbf{R}^3)$ (also bounded in $\mathrm{L}^3(\Omega;\mathbf{R}^3)$)? Yes! (Tartar, 1990)

$$\mathbf{M} = \frac{1}{\nu}\left\langle\!\left\langle \boldsymbol{\mu}, (\mathsf{v}^2 - (\mathsf{v}\cdot\boldsymbol{\xi})^2)\boldsymbol{\xi}\otimes\boldsymbol{\xi}\right\rangle\!\right\rangle \ .$$

Note. The meaning of the formula: $(\forall\,\varphi \in \mathrm{C}_c^\infty(\Omega))$

$$\int_\Omega \mathbf{M}(\mathbf{x})\varphi(\mathbf{x})\,d\mathbf{x} = \frac{1}{\nu}\left[\langle \mathrm{tr}\boldsymbol{\mu}, \varphi\boxtimes(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\rangle - \langle \boldsymbol{\mu}, \varphi\boxtimes(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\otimes(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\rangle\right] \ .$$

$\mathbf{M}$ is not only a measure, but a function.

What in the time-dependent case?

Stationary model motivated the introduction of H-measures.
Time-dependent led to a variant.
Tartar with Chun Liu and Konstantina Trevisa some thirty years ago; only written record in Multiscales 2000.
M. Lazar and myself — wrote it down (technical difference in the scaling).

Time dependent case

On $\mathbf{R}^3$ (we need good estimates for the pressure).

Tartar's model from 1985:

$$\begin{cases} \partial_t \mathsf{u}_n - \nu \triangle \mathsf{u}_n + \mathsf{u}_n \times \operatorname{rot}(\mathsf{v}_0 + \lambda \mathsf{v}_n) + \nabla p_n = \mathsf{f}_n \\ \operatorname{div} \mathsf{u}_n = 0 \ . \end{cases}$$

Assume that

$$\begin{aligned} \mathsf{u}_n &\longrightarrow \mathsf{u}_0 \quad \text{in } \mathrm{L}^2([0,T];\mathrm{H}^1(\mathbf{R}^3;\mathbf{R}^3)) \ , \\ \mathsf{u}_n &\overset{*}{\longrightarrow} \mathsf{u}_0 \quad \text{in } \mathrm{L}^\infty([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3)) \ . \end{aligned}$$

and (p_n) is bounded in $\mathrm{L}^2([0,T] \times \mathbf{R}^3)$.
Oscillation in (v_n) generates oscillation in $(\nabla \mathsf{u}_n)$, which dissipates energy via viscosity.
This should be visible from macroscopic equation (equation satisfied by u_0).

Sufficient assumptions on v_n and f_n

$\mathsf{f}_n = \operatorname{div} \mathbf{G}_n$, with $\mathbf{G}_n \longrightarrow \mathbf{G}_0$ in $\mathrm{L}^2([0,T] \times \mathbf{R}^3; \mathrm{M}_{3\times 3})$

$\mathsf{v}_0 \in \mathrm{L}^2([0,T]; \mathrm{L}^\infty(\mathbf{R}^3; \mathbf{R}^3)) + \mathrm{L}^\infty([0,T]; \mathrm{L}^3(\mathbf{R}^3; \mathbf{R}^3))$

$\mathsf{v}_n = \mathsf{a}_n + \mathsf{b}_n$, where

$\mathsf{a}_n \xrightarrow{\;*\;} 0$ in $\mathrm{L}^q([0,T]; \mathrm{L}^\infty(\mathbf{R}^3; \mathbf{R}^3))$, for some $q > 2$,

$\mathsf{b}_n \xrightarrow{\;*\;} 0$ in $\mathrm{L}^\infty([0,T]; \mathrm{L}^r(\mathbf{R}^3; \mathbf{R}^3))$, for some $r > 3$.

Homogenised equation

Theorem. There is a subsequence and a function $\mathbf{M} \geqslant \mathbf{0}$ such that the limit u_0 satisfies:

$$\begin{cases} \partial_t \mathsf{u}_0 - \nu \triangle \mathsf{u}_0 + \mathsf{u}_0 \times \operatorname{rot} \mathsf{v}_0 + \lambda^2 \mathbf{M} \mathsf{u}_0 + \nabla p_0 = \mathsf{f}_0 \\ \operatorname{div} \mathsf{u}_0 = 0 \,, \end{cases}$$

and that we have the convergence

$$\nu |\nabla \mathsf{u}_n|^2 \longrightarrow \nu |\nabla \mathsf{u}_0|^2 + \lambda^2 \mathbf{M} \mathsf{u}_0 \cdot \mathsf{u}_0 \qquad \text{in } \mathcal{D}'(\mathbf{R}^{1+3}) \,.$$

There is a new term, $\mathbf{M}$, in the macroscopic equation.
How can it be computed?

Oscillating test functions

$$\begin{cases} -\partial_t \mathsf{w}_n - \nu \triangle \mathsf{w}_n + \mathsf{k} \times \operatorname{rot} \mathsf{v}_n + \nabla r_n = 0 \\ \operatorname{div} \mathsf{w}_n = 0 \ , \end{cases}$$

supplemented by requirements:

$\mathsf{w}_n \longrightarrow 0$ in $\mathrm{L}^2([0,T];\mathrm{H}^1(\mathbf{R}^3;\mathbf{R}^3))$, and
$\mathsf{w}_n \overset{*}{\longrightarrow} 0$ in $\mathrm{L}^\infty([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3))$.

Sufficient to take homogeneous condition at $t = T$,
and (additional assumption) v_n bounded in $\mathrm{L}^2([0,T];\mathrm{L}^2(\mathbf{R}^3;\mathbf{R}^3))$.
This in particular gives r_n bounded in $\mathrm{L}^2([0,T] \times \mathbf{R}^3)$.

$$\nu \int_{\mathbf{R}^{1+3}} \varphi |\nabla \mathsf{w}_n|^2 \, d\mathbf{y} \longrightarrow \int_{\mathbf{R}^{1+3}} \varphi \mathbf{M}\mathsf{k} \cdot \mathsf{k} \, d\mathbf{y} \ ,$$

$\mathbf{M} \in \mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^3;\mathrm{M}_{3\times 3\times)})$ and $\langle \mathbf{M}\mathsf{k} \mid \mathsf{k} \rangle \geqslant 0, \quad \mathsf{k} \in \mathbf{R}^3$.

Can we avoid w_n?

Theorem. Let $\boldsymbol{\mu}$ be a parabolic H-measure associated to a subsequence of (v_n).

$$\int_{\mathbf{R}^{1+3}} \mathbf{M}(t,\mathbf{x})\phi(t,\mathbf{x})dtd\mathbf{x} =$$

$$= 4\pi^2\nu\Big\langle\Big(\mathsf{tr}\boldsymbol{\mu}|\boldsymbol{\xi}|^2 - \boldsymbol{\mu}\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\xi})\Big)\frac{(\boldsymbol{\xi}\otimes\boldsymbol{\xi})}{\tau^2+\nu^2 4\pi^2|\boldsymbol{\xi}|^4}, \phi\boxtimes 1\Big\rangle,$$

with $\phi \in \mathrm{C}_c^\infty(\langle 0,T\rangle\times\mathbf{R}^3)$.

Proof.

For w_n we have (with $0 \leqslant \mathbf{M} \in \mathrm{L}^2([0,T];\mathrm{H}^{-1}(\mathbf{R}^3;\mathrm{M}_{3\times 3})))$):

$$\nu \int_{\mathbf{R}^{1+3}} \varphi |\nabla \mathsf{w}_n|^2 \, d\mathbf{y} \longrightarrow \int_{\mathbf{R}^{1+3}} \varphi \mathbf{M} \mathsf{k} \cdot \mathsf{k} \, d\mathbf{y} \ .$$

From estimates on r_n and v_n we get $\mathsf{w}_n' \longrightarrow 0$ in $\mathrm{L}^2(0,T;\mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^3))$, and compactness lemma gives us $\mathsf{w}_n \to 0$ in $\mathrm{L}^2_{\mathrm{loc}}([0,T] \times \mathbf{R}^3)$.

Therefore:

$$\lim_n \int_{\mathbf{R}^{1+3}} |\varphi \nabla \mathsf{w}_n|^2 \, d\mathbf{y} = \lim_n \int_{\mathbf{R}^{1+3}} |\nabla(\varphi \mathsf{w}_n)|^2 \, d\mathbf{y} \ .$$

Localise ...

Localise by multiplying the auxilliary problem by $\varphi \in \mathrm{C}_c^\infty(\langle 0,T\rangle \times \mathbf{R}^3)$

$$-\partial_t(\varphi \mathsf{w}_n) - \nu\triangle(\varphi \mathsf{w}_n) + \mathsf{k} \times \mathsf{rot}\,(\varphi \mathsf{v}_n) = -\nabla(\varphi r_n) + \mathsf{q}_n \ ,$$

$$\mathsf{q}_n = -(\partial_t \varphi)\mathsf{w}_n - \nu(\triangle\varphi)\mathsf{w}_n - 2\nu(\nabla \mathsf{w}_n)\nabla\varphi + \mathsf{k} \times (\nabla\varphi \times \mathsf{v}_n) + r_n\nabla\varphi \ ,$$

$\mathsf{q}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^{1+3})$ (and also strongly in $\mathrm{H}^{-\frac{1}{2},-1}(\mathbf{R}^{1+3})$).
As $\mathsf{w}_n \longrightarrow 0$ in $\mathrm{L}^2([0,T];\mathrm{H}^1(\mathbf{R}^3))$, so localised w_n and $\nabla \mathsf{w}_n$ converge weakly in L^2.

Of course, localised v_n and r_n converge weakly in L^2 as well.
From boundedness of the support of φ, we have strong convergence in $\mathrm{H}^{-\frac{1}{2},-1}$.

The Fourier transform

$$(-2\pi i\tau + \nu 4\pi^2 \boldsymbol{\xi}^2)\widehat{\varphi \mathsf{w}_n} = -\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n} \Big) - 2\pi i \widehat{\varphi r_n} \boldsymbol{\xi} + \hat{\mathsf{q}}_n \ ,$$

and dividing by $(-2\pi i\tau + \nu 4\pi^2 \boldsymbol{\xi}^2)$ we get

$$\widehat{\varphi \mathsf{w}_n} = \frac{-\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n} \Big) - 2\pi i \widehat{\varphi r_n} \boldsymbol{\xi} + \hat{\mathsf{q}}_n}{-2\pi i\tau + \nu 4\pi^2 \boldsymbol{\xi}^2} \ .$$

The penultimate term disappears if we project it to the plane $\perp \boldsymbol{\xi}$ (projection $P_{\boldsymbol{\xi}}$).

$\operatorname{div} \mathsf{w}_n = 0$, so $\boldsymbol{\xi} \cdot \hat{\mathsf{w}}_n = 0$; which does not hold for $\operatorname{div}(\varphi \mathsf{w}_n) = \nabla\varphi \cdot \mathsf{w}_n$.
However, the RHS converges strongly in L^2 to 0, so in the Fourier space:

$$2\pi \boldsymbol{\xi} \cdot \widehat{\varphi \mathsf{w}_n} \longrightarrow 0 \ .$$

Projection by $P_{\boldsymbol{\xi}}$

After projection

$$\widehat{\varphi \mathsf{w}_n} = \frac{-P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big)\Big) + P_{\boldsymbol{\xi}} \hat{\mathsf{q}}_n}{-2\pi i \tau + \nu 4\pi^2 \boldsymbol{\xi}^2} + \mathsf{d}_n \ ,$$

with $\mathsf{d}_n \longrightarrow 0$ in L^2.
By Plancherel

$$\begin{aligned}
\lim_n \int_\Omega \nu |\nabla(\varphi \mathsf{w}_n)|^2 \, d\mathbf{x} &= \lim_n \int_{\mathbf{R}}^{1+d} \nu 4\pi^2 |\widehat{(\varphi \mathsf{w}_n)}|^2 d\tau d\boldsymbol{\xi} \\
&= \lim_n \int_{\mathbf{R}}^{1+d} \nu 4\pi^2 \boldsymbol{\xi}^2 \left| \frac{P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big) + \hat{\mathsf{q}}_n\Big)}{-2\pi i \tau + \nu 4\pi^2 \boldsymbol{\xi}^2} \right|^2 d\tau d\boldsymbol{\xi} \\
&= \lim_n \int_{\mathbf{R}}^{1+d} \nu \boldsymbol{\xi}^2 \frac{\left| P_{\boldsymbol{\xi}}\Big(\mathsf{k} \times \Big((2\pi i \boldsymbol{\xi}) \times \widehat{\varphi \mathsf{v}_n}\Big) + \hat{\mathsf{q}}_n\Big) \right|^2}{\tau^2 + \nu 4\pi^2 \boldsymbol{\xi}^4} d\tau d\boldsymbol{\xi}
\end{aligned}$$

Applying the Lemma (analysis)

$$\frac{|\boldsymbol{\xi}|\hat{\mathsf{q}}_n}{\sqrt{\tau^2 + \nu 4\pi^2 \boldsymbol{\xi}^4}} \to 0 \quad \text{in} \qquad \mathrm{L}^2(\mathbf{R}^{1+3}) \ .$$

By $P_{\boldsymbol{\eta}}$

$$\left| P_{\boldsymbol{\eta}}\Big(\mathsf{k} \times (\boldsymbol{\eta} \times \mathbf{a})\Big)\right|^2 = (\mathsf{k} \cdot \boldsymbol{\eta})^2 \Big(|\mathsf{a}|^2 - |\mathsf{a} \cdot \boldsymbol{\eta}_0|^2\Big)$$

where $\boldsymbol{\eta}_0$ is the unit vector in the direction of $\boldsymbol{\eta}$.
Note that k and $\boldsymbol{\eta}$ are real, while only a is complex. Therefore:

$$\lim_n \int_\Omega \nu |\nabla(\varphi \mathsf{w}_n)|^2 \, d\mathbf{x}$$

$$= \lim_n \int_{\mathbf{R}^3} \boldsymbol{\xi}^2 \frac{\Big(\mathsf{k} \cdot 2\pi i \boldsymbol{\xi}\Big)^2 \Big(|\widehat{\varphi \mathsf{v}_n}|^2 - \Big|\widehat{\varphi \mathsf{v}_n} \cdot \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big|^2\Big)}{\tau^2 + \nu 4\pi^2 \boldsymbol{\xi}^4} \, d\boldsymbol{\xi} \ .$$

Finally (after some algebra)

$$\lim_n \int_{\mathbf{R}^3} \boldsymbol{\xi}_0^2 \frac{\left(\mathsf{k}\cdot 2\pi i \boldsymbol{\xi}_0\right)^2 \left(|\widehat{\varphi \mathsf{v}_n}|^2 - \left|\widehat{\varphi \mathsf{v}_n}\cdot \frac{\boldsymbol{\xi}_0}{|\boldsymbol{\xi}_0|}\right|^2\right)}{\tau_0^2 + \nu 4\pi^2 \boldsymbol{\xi}_0^4}\, d\boldsymbol{\xi} =$$

$$= \frac{1}{\nu}\langle \mathsf{tr}\boldsymbol{\mu}, (\frac{\boldsymbol{\xi}_0\cdot \mathsf{k}}{\tau_0^2 + \nu 4\pi^2 \boldsymbol{\xi}_0^4})^2 \varphi\overline{\varphi}\rangle$$

$$- \frac{1}{\nu}\langle \boldsymbol{\mu}, (\frac{\boldsymbol{\xi}_0\cdot \mathsf{k}}{\tau_0^2 + \nu 4\pi^2 \boldsymbol{\xi}_0^4})^2 \varphi\overline{\varphi}\boldsymbol{\xi}\otimes\boldsymbol{\xi}\rangle \ .$$

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega; \mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exist a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}_{\mathrm{b}}(\Omega \times \mathbf{R}^d; \mathrm{M}_{\mathrm{r}}(\mathbf{C}))$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big(\widehat{\varphi_1 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \otimes \widehat{\varphi_2 \mathsf{u}_{n'}}(\boldsymbol{\xi}) \Big) \psi(\omega_{n'} \boldsymbol{\xi})\, d\boldsymbol{\xi} = \Big\langle \boldsymbol{\mu}_{sc}^{(\omega_n)}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \Big\rangle .$$

Measure $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ *we call the semiclassical measure with characteristic length* (ω_n) *corresponding to the (sub)sequence* (u_n).

Definition (u_n) is (ω_n)-oscillatory if
$(\forall\, \varphi \in \mathrm{C}_c^\infty(\Omega)) \quad \lim_{R\to\infty} \limsup_n \int_{|\boldsymbol{\xi}| \geqslant \frac{R}{\omega_n}} |\widehat{\varphi \mathsf{u}_n}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} = 0\,.$

Theorem.

$$\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{sc}^{(\omega_n)} = \mathbf{0} \quad \& \quad (\mathsf{u}_n) \textit{ is } (\omega_n) - \textit{oscillatory}\,.$$

Localisation principle for semiclassical measures

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\mathbf{P}_n \mathsf{u}_n := \sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega \,,$$

where

- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$.

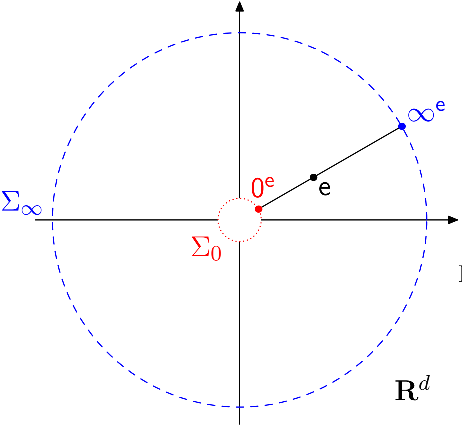
Then we have

$$\mathbf{p} \boldsymbol{\mu}_{sc}^{\top} = \mathbf{0} \,,$$

where $\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}| \leqslant m} \boldsymbol{\xi}^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})$, and $\boldsymbol{\mu}_{sc}$ is semiclassical measure with characteristic length (ε_n), corresponding to (u_n).

Problem: $\boldsymbol{\mu}_{sc} = \mathbf{0}$ is not enough for the strong convergence!

Compatification of $\mathbf{R}^d \setminus \{0\}$



$$\Sigma_0 := \{0^{\mathsf{e}} \;:\; \mathsf{e} \in \mathrm{S}^{d-1}\}$$

$$\Sigma_\infty := \{\infty^{\mathsf{e}} \;:\; \mathsf{e} \in \mathrm{S}^{d-1}\}$$

$$\mathrm{K}_{0,\infty}(\mathbf{R}^d) := \mathbf{R}^d \setminus \{0\} \cup \Sigma_0 \cup \Sigma_\infty$$

Corollary. *a)* $\mathrm{C}_0(\mathbf{R}^d) \subseteq \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
b) $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\psi \circ \boldsymbol{\pi} \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, *where* $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$. ∎

One-scale H-measures

Theorem. *If* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, $\omega_n \to 0^+$, *then there exists a subsequence* $(\mathsf{u}_{n'})$ *and* $\boldsymbol{\mu}_{sc}^{(\omega_n)} \in \mathcal{M}_{\rm b}(\Omega\times\mathbf{R}^d;\mathrm{M}_{\rm r}(\mathbf{C}))$ *such that for any* $\varphi_1,\varphi_2 \in \mathrm{C}_c^\infty(\Omega)$ *and* $\psi \in \mathcal{S}(\mathbf{R}^d)$

$$\lim_{n'} \int_{\mathbf{R}^d} \Big((\widehat{\varphi_1 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \otimes (\widehat{\varphi_2 \mathsf{u}_{n'}})(\boldsymbol{\xi}) \Big) \psi(\omega_{n'}\boldsymbol{\xi})\, d\boldsymbol{\xi} = \Big\langle \boldsymbol{\mu}_{sc}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \Big\rangle .$$

Measure $\boldsymbol{\mu}_{sc}^{(\omega_n)}$ *is called the semiclassical measure with characteristic length* (ω_n) *corresponding to the (sub)sequence* (u_n). ∎

Luc Tartar: *The general theory of homogenization: A personalized introduction, Springer*, 2009.

Luc Tartar: *Multi-scale H-measures, Discrete and Continuous Dynamical Systems, S* **8** (2015) 77–90.

N. A., Marko Erceg, Martin Lazar: *Localisation principle for one-scale H-measures, Journal of Functional Analysis* **272** (2017) 3410–3454.

Marko Erceg, Martin Lazar: *Characteristic scales of bounded* L^2 *sequences, Asymptotic Analysis* **109** (2018) 171–192.

Idea of the proof

Tartar's approach:

- $\mathsf{v}_n(\mathbf{x}, x^{d+1}) := \mathsf{u}_n(\mathbf{x}) e^{\frac{2\pi i x^{d+1}}{\omega_n}} \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega \times \mathbf{R}; \mathbf{C}^r)$
- $\boldsymbol{\nu}_H \in \mathcal{M}(\Omega \times \mathbf{R} \times \mathrm{S}^d; \mathrm{M_r}(\mathbf{C}))$
- $\boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}$ is obtained from $\boldsymbol{\nu}_H$ (suitable projection in x^{d+1} and ξ_{d+1})

Our approach:

- First commutation lemma:

Lemma. *Let* $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, *and denote* $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. *Then the commutator can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K\,,$$

where K *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$, *while* $\tilde{C}_n \longrightarrow 0$ *in the operator norm on* $\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d))$. ∎

- standard procedure: (a variant of) the kernel theorem, separability, . . .

Some properties of $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$

Theorem.

a) $\boldsymbol{\mu}^*_{\mathrm{K}_{0,\infty}} = \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$, $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} \geqslant \mathbf{0}$

b) $\mathsf{u}_n \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}} 0 \iff \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}$

c) $\mathrm{tr}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}(\Omega \times \Sigma_\infty) = 0 \iff (\mathsf{u}_n)$ *is* $(\omega_n) -$ *oscillatory*

∎

Theorem. $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$, $\psi \in \mathrm{C}_0(\mathbf{R}^d)$, $\tilde\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, $\omega_n \to 0^+$,

a) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle = \langle \boldsymbol{\mu}^{(\omega_n)}_{sc}, \varphi_1\bar\varphi_2 \boxtimes \psi\rangle$,

b) $\langle \boldsymbol{\mu}^{(\omega_n)}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi\circ\boldsymbol{\pi}\rangle = \langle \boldsymbol{\mu}_H, \varphi_1\bar\varphi_2 \boxtimes \tilde\psi\rangle$,

where $\boldsymbol{\pi}(\boldsymbol{\xi}) = \boldsymbol{\xi}/|\boldsymbol{\xi}|$.

∎

Localisation principle

Let $\Omega \subseteq \mathbf{R}^d$ open, $m \in \mathbf{N}$, $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

where

- $l \in 0..m$
- $\varepsilon_n \to 0^+$
- $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M_r}(\mathbf{C}))$
- $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ such that

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + \sum_{s=l}^m \varepsilon_n^{s-l} |\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \qquad (\mathrm{C}(\varepsilon_n))$$

Lemma. *a) $(\mathrm{C}(\varepsilon_n))$ is equivalent to*

$$(\forall \varphi \in \mathrm{C}^\infty_c(\Omega)) \qquad \frac{\widehat{\varphi \mathsf{f}_n}}{1 + |\boldsymbol{\xi}|^l + \varepsilon_n^{m-l} |\boldsymbol{\xi}|^m} \longrightarrow 0 \quad \textit{in} \quad \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)\,.$$

b) $(\exists\, k \in l..m)$ $\mathsf{f}_n \longrightarrow 0$ *in* $\mathrm{H}^{-k}_{\mathrm{loc}}(\Omega; \mathbf{C}^r) \quad \Longrightarrow \quad (\varepsilon_n^{k-l} \mathsf{f}_n)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$. ∎

Localisation principle

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,. \qquad (\mathrm{C}(\varepsilon_n))$$

Theorem. [Tartar (2009)] *Under previous assumptions and* $l=1$, *one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ε_n) *corresponding to* (u_n) *satisfies*

$$\operatorname{supp}(\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top) \subseteq \Omega\times\Sigma_0\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{1\leqslant|\boldsymbol{\alpha}|\leqslant m}(2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

Theorem. [N.A., Erceg, Lazar (2017)] *Under previous assumptions, one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ε_n) *corresponding to* (u_n) *satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^\top = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m}(2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

Localisation principle - final generalisation

Theorem. *Take* $\varepsilon_n > 0$ *bounded,* $\mathsf{u}_n \rightharpoonup 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *and*

$$\sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l} \partial_{\boldsymbol{\alpha}}(\mathbf{A}_n^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $\mathbf{A}_n^{\boldsymbol{\alpha}} \longrightarrow \mathbf{A}^{\boldsymbol{\alpha}}$ *uniformly on compact sets, and* $\mathsf{f}_n \in \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ *satisfies* $(\mathrm{C}(\varepsilon_n))$.
Then for $\omega_n \to 0^+$ *such that* $c := \lim_n \frac{\varepsilon_n}{\omega_n} \in [0, \infty]$, *the corresponding one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ω_n) *satisfies*

$$\mathbf{p} \boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}^{\top} = \mathbf{0} \,,$$

where

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \begin{cases} \sum_{|\boldsymbol{\alpha}|=l} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = 0 \\ \sum_{l \leqslant |\boldsymbol{\alpha}| \leqslant m} (2\pi i c)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c \in \langle 0, \infty \rangle \\ \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l + |\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) & , \quad c = \infty \end{cases}$$

Moreover, if there exists $\varepsilon_0 > 0$ *such that* $\varepsilon_n > \varepsilon_0$, $n \in \mathbf{N}$, *we can take*

$$\mathbf{p}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}|=m} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^m} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x}) \,.$$

∎

As a corollary from the previous theorem we can derive localisation principles for H-measures and semiclassical measures.

Thank you for your attention.