Homogenisation and microlocal energy propagation for the wave equation revisited

Nenad Antonić

Department of Mathematics
Faculty of Science
University of Zagreb

International Conference on Generalized Functions
Wien, September 21$^{\mathrm{st}}$, 2022

http://riemann.math.hr/mitpde/

Microlocal defect functionals

- Overview
- Test functions in the dual space
- Kernel theorem
- One-scale H-distributions
- Localisation principle

The wave equation

- Kirchhoff-Love plate theory
- Homogenisation of Kirchhoff-Love plates
- Small-amplitude homogenisation for plates
- Comparison to the periodic case
- Homogenisation of the vibrating plate equation
- Propagation property

Microlocal defect functionals

H-measures vs. defect measures

H-measures or microlocal defect measures represent a generalisation of defect measures. Besides the space variables, they depend on the dual variables as well.

H-measures vs. defect measures

H-measures or microlocal defect measures represent a generalisation of defect measures. Besides the space variables, they depend on the dual variables as well. An H-measure is a Radon measure on the cospherical bundle

$$\Omega \times \mathrm{S}^{d-1} \subseteq T^*\Omega \simeq \Omega \times \mathbf{R}^d$$

over a domain $\Omega \subseteq \mathbf{R}^d$, and it is associated to a weakly converging sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$.

H-measures vs. defect measures

H-measures or microlocal defect measures represent a generalisation of defect measures. Besides the space variables, they depend on the dual variables as well. An H-measure is a Radon measure on the cospherical bundle

$$\Omega \times \mathrm{S}^{d-1} \subseteq T^*\Omega \simeq \Omega \times \mathbf{R}^d$$

over a domain $\Omega \subseteq \mathbf{R}^d$, and it is associated to a weakly converging sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$.

Consider a plain wave:

$$u_n(\mathbf{x}) = \varphi(\mathbf{x}) e^{2\pi i \frac{\mathbf{x}}{\varepsilon_n} \cdot \mathsf{k}},$$

where $\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, $\mathsf{k} \in \mathbf{R}^d \setminus \{0\}$, and $\varepsilon_n \to 0^+$. This sequence weakly converges in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$ to 0 (but not strongly, except in the trivial case $\varphi = 0$).

H-measures vs. defect measures

H-measures or microlocal defect measures represent a generalisation of defect measures. Besides the space variables, they depend on the dual variables as well. An H-measure is a Radon measure on the cospherical bundle

$$\Omega \times \mathrm{S}^{d-1} \subseteq T^*\Omega \simeq \Omega \times \mathbf{R}^d$$

over a domain $\Omega \subseteq \mathbf{R}^d$, and it is associated to a weakly converging sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$.

Consider a plain wave:

$$u_n(\mathbf{x}) = \varphi(\mathbf{x}) e^{2\pi i \frac{\mathbf{x}}{\varepsilon_n} \cdot \mathsf{k}},$$

where $\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, $\mathsf{k} \in \mathbf{R}^d \setminus \{0\}$, and $\varepsilon_n \to 0^+$. This sequence weakly converges in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$ to 0 (but not strongly, except in the trivial case $\varphi = 0$).

Defect measure is the limit of $|u_n|^2 = |\varphi|^2$ in the space of (unbounded) Radon measures with respect to the weak-$*$ topology — $|\varphi|^2 \lambda^d$.

H-measures vs. defect measures

H-measures or microlocal defect measures represent a generalisation of defect measures. Besides the space variables, they depend on the dual variables as well. An H-measure is a Radon measure on the cospherical bundle

$$\Omega \times \mathrm{S}^{d-1} \subseteq T^*\Omega \simeq \Omega \times \mathbf{R}^d$$

over a domain $\Omega \subseteq \mathbf{R}^d$, and it is associated to a weakly converging sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$.

Consider a plain wave:

$$u_n(\mathbf{x}) = \varphi(\mathbf{x}) e^{2\pi i \frac{\mathbf{x}}{\varepsilon_n} \cdot \mathsf{k}},$$

where $\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, $\mathsf{k} \in \mathbf{R}^d \setminus \{0\}$, and $\varepsilon_n \to 0^+$. This sequence weakly converges in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$ to 0 (but not strongly, except in the trivial case $\varphi = 0$).

Defect measure is the limit of $|u_n|^2 = |\varphi|^2$ in the space of (unbounded) Radon measures with respect to the weak-$*$ topology — $|\varphi|^2 \lambda^d$.

On the other hand, the H-measure is

$$|\varphi|^2 \lambda^d \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}},$$

where $\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$ (the Dirac measure at point $\mathsf{k}/|\mathsf{k}|$) is a measures in the dual space (variable $\boldsymbol{\xi}$).

H-measures vs. defect measures

H-measures or microlocal defect measures represent a generalisation of defect measures. Besides the space variables, they depend on the dual variables as well. An H-measure is a Radon measure on the cospherical bundle

$$\Omega \times \mathrm{S}^{d-1} \subseteq T^*\Omega \simeq \Omega \times \mathbf{R}^d$$

over a domain $\Omega \subseteq \mathbf{R}^d$, and it is associated to a weakly converging sequence in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$.

Consider a plain wave:

$$u_n(\mathbf{x}) = \varphi(\mathbf{x}) e^{2\pi i \frac{\mathbf{x}}{\varepsilon_n} \cdot \mathsf{k}},$$

where $\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$, $\mathsf{k} \in \mathbf{R}^d \setminus \{0\}$, and $\varepsilon_n \to 0^+$. This sequence weakly converges in $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$ to 0 (but not strongly, except in the trivial case $\varphi = 0$).

Defect measure is the limit of $|u_n|^2 = |\varphi|^2$ in the space of (unbounded) Radon measures with respect to the weak-$*$ topology — $|\varphi|^2 \lambda^d$.

On the other hand, the H-measure is

$$|\varphi|^2 \lambda^d \boxtimes \delta_{\frac{\mathsf{k}}{|\mathsf{k}|}},$$

where $\delta_{\frac{\mathsf{k}}{|\mathsf{k}|}}$ (the Dirac measure at point $\mathsf{k}/|\mathsf{k}|$) is a measures in the dual space (variable $\boldsymbol{\xi}$).

Hence, the direction of oscillation is inherent in the H-measure.

H-measures vs. semiclassical (Wigner) measures

In the example the H-measure does not distinguish between sequences with different frequencies $\frac{1}{\varepsilon_n}$. We need to incorporate a scale.
Semiclassical measures are Radon measures on the cotangential bundle $\Omega \times \mathbf{R}^d$. They depend upon a characteristic length (ω_n), $\omega_n \to 0^+$ — more suitable where such a characteristic length naturally appears (e.g. in highly oscillating problems for partial differential equations).

H-measures vs. semiclassical (Wigner) measures

In the example the H-measure does not distinguish between sequences with different frequencies $\frac{1}{\varepsilon_n}$. We need to incorporate a scale.
Semiclassical measures are Radon measures on the cotangential bundle $\Omega \times \mathbf{R}^d$. They depend upon a characteristic length (ω_n), $\omega_n \to 0^+$ — more suitable where such a characteristic length naturally appears (e.g. in highly oscillating problems for partial differential equations).

The scale brings new issues: if (ω_n) is chosen inappropriately, we can lose information.

H-measures vs. semiclassical (Wigner) measures

In the example the H-measure does not distinguish between sequences with different frequencies $\frac{1}{\varepsilon_n}$. We need to incorporate a scale.
Semiclassical measures are Radon measures on the cotangential bundle $\Omega \times \mathbf{R}^d$. They depend upon a characteristic length (ω_n), $\omega_n \to 0^+$ — more suitable where such a characteristic length naturally appears (e.g. in highly oscillating problems for partial differential equations).

The scale brings new issues: if (ω_n) is chosen inappropriately, we can lose information.

For example, if $\lim_n \frac{\omega_n}{\varepsilon_n} = +\infty$, the semiclassical measure associated to the plane wave is equal to zero measure. This in particular implies that, in contrast to H-measures, a zero semiclassical measure does not necessarily guarantee the strong convergence of the associated sequence (the so-called (ω_n)-oscillatory property needs to be satisfied as well).

H-measures vs. semiclassical (Wigner) measures

In the example the H-measure does not distinguish between sequences with different frequencies $\frac{1}{\varepsilon_n}$. We need to incorporate a scale.
Semiclassical measures are Radon measures on the cotangential bundle $\Omega \times \mathbf{R}^d$. They depend upon a characteristic length (ω_n), $\omega_n \to 0^+$ — more suitable where such a characteristic length naturally appears (e.g. in highly oscillating problems for partial differential equations).

The scale brings new issues: if (ω_n) is chosen inappropriately, we can lose information.

For example, if $\lim_n \frac{\omega_n}{\varepsilon_n} = +\infty$, the semiclassical measure associated to the plane wave is equal to zero measure. This in particular implies that, in contrast to H-measures, a zero semiclassical measure does not necessarily guarantee the strong convergence of the associated sequence (the so-called (ω_n)-oscillatory property needs to be satisfied as well).

H-measures and semiclassical measures are in a general relation (neither is a generalisation of the other) and either has some advantages and disadvantages.

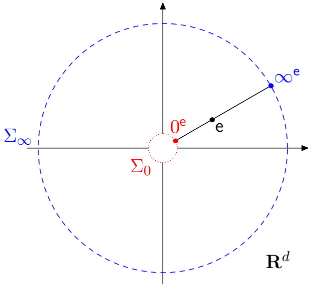
One-scale H-measures

One-scale H-measures are a true extension of H-measures and semiclassical measures.

One-scale H-measures

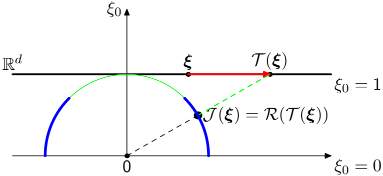
One-scale H-measures are a true extension of H-measures and semiclassical measures.

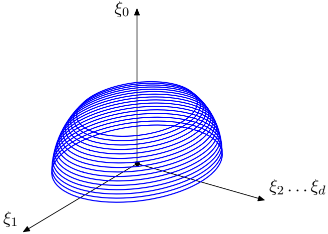
Crucial is a proper choice of the domain for dual variables.
For a fine tuning with characteristic lengths the set has to be *thick enough*, but we need to allow for directions to be *detected* also at the origin and infinity.
This can be achieved with a radial compactification of $\mathbf{R}^d \setminus \{0\}$, denoted by $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$, which is homeomorphic to the d-dimensional spherical shell.



Compactification of $\mathbf{R}^d_* = \mathbf{R}^d \setminus \{0\}$

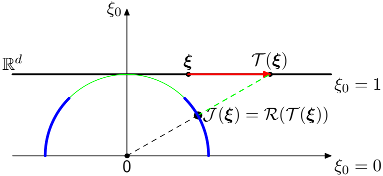
$$\mathcal{J} := \mathcal{R} \circ \mathcal{T} : \mathbf{R}^d_* \to \mathrm{S}^d_{(0,r_1)}$$

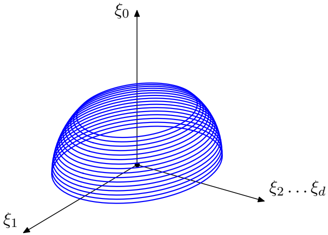




Compactification of $\mathbf{R}_*^d = \mathbf{R}^d \setminus \{0\}$

$$\mathcal{J} := \mathcal{R} \circ \mathcal{T} : \mathbf{R}_*^d \to \mathrm{S}^d_{(0,r_1)}$$

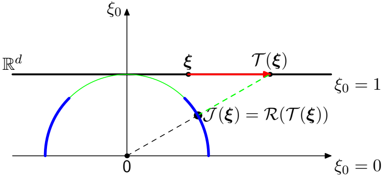


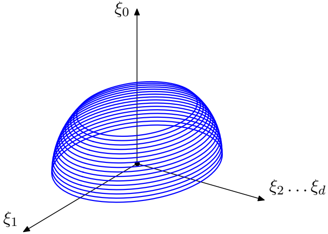


$$\mathcal{R}(\mathbf{R}^d \setminus \mathrm{K}[0, r_0]) = \left\{ (\zeta_0, \boldsymbol{\zeta}) \in \mathrm{S}^d : 0 < \zeta_0 < r_1 \right\} =: \mathrm{S}^d_{(0,r_1)}$$

Compactification of $\mathbf{R}^d_* = \mathbf{R}^d \setminus \{0\}$

$$\mathcal{J} := \mathcal{R} \circ \mathcal{T} : \mathbf{R}^d_* \to \mathrm{S}^d_{(0,r_1)}$$





$$\mathcal{R}(\mathbf{R}^d \setminus \mathrm{K}[0, r_0]) = \left\{(\zeta_0, \boldsymbol{\zeta}) \in \mathrm{S}^d : 0 < \zeta_0 < r_1\right\} =: \mathrm{S}^d_{(0,r_1)}$$

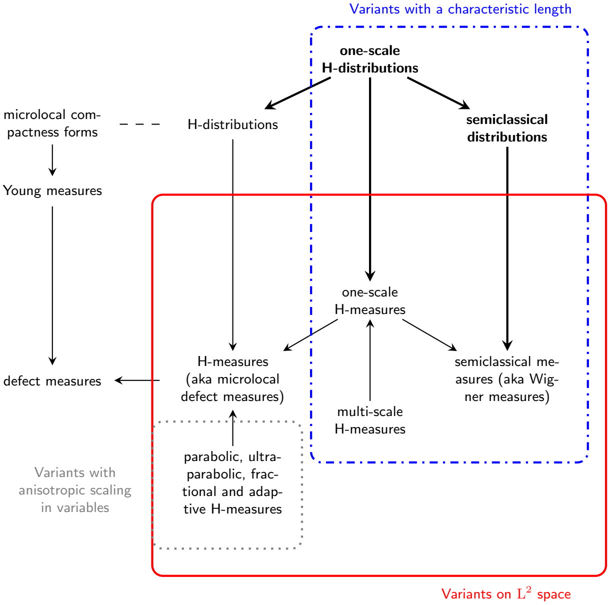
and its closure in the ambient euclidean space $\mathbf{R}^{1+d}$

$$\mathrm{Cl}\, \mathrm{S}^d_{(0,r_1)} = \left\{(\zeta_0, \boldsymbol{\zeta}) \in \mathrm{S}^d : 0 \leqslant \zeta_0 \leqslant r_1\right\} =: \mathrm{S}^d_{[0,r_1]} \;,$$

which is diffeomorphic to the compactification $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

to Fourier multipliers

Overview of MDF



to Existence theorem

Compactification of $\mathbf{R}^d_* = \mathbf{R}^d \setminus \{0\}$

For the compactifying map $\mathcal{J}$ we take the composition of the translation from the origin in the radial direction for $r_0 > 0$:

$$\mathbf{R}^d_* \ni \boldsymbol{\xi} \overset{\mathcal{T}}{\longmapsto} \frac{|\boldsymbol{\xi}| + r_0}{|\boldsymbol{\xi}|}\boldsymbol{\xi} \in \mathbf{R}^d \setminus \mathrm{K}[0, r_0] \ ,$$

and a compactifying map of the radial compactification.
For the latter, we first identify $\mathbf{R}^d$ with the hypersurface $\xi_0 = 1$ in $\mathbf{R}^{1+d}_{\xi_0,\boldsymbol{\xi}}$, and then apply the modified stereographic projection based on the line through the origin (instead of the South Pole). More precisely, the radial compactification map $\mathcal{R}$ maps $\boldsymbol{\xi}$ to the intersection of $[0,1] \ni t \mapsto (t, t\boldsymbol{\xi})$ (the line through $(1, \boldsymbol{\xi})$ and $(0, 0)$ in $\mathbf{R}^{1+d}$) and the upper half of the unit sphere centred at the origin: $\mathrm{S}^d_+ := \{(\zeta_0, \boldsymbol{\zeta}) \in \mathrm{S}^d : \zeta_0 > 0\}$. Since the intersection occurs at $t = (1 + |\boldsymbol{\xi}|^2)^{-\frac{1}{2}}$, we have that $\mathcal{R} : \mathbf{R}^d \to \mathrm{S}^d_+$ is given by

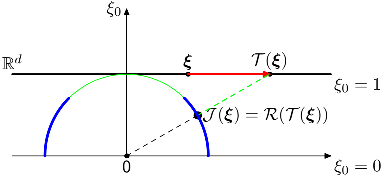
$$\mathcal{R}(\boldsymbol{\xi}) = \Big(\frac{1}{\sqrt{1 + |\boldsymbol{\xi}|^2}}, \frac{\boldsymbol{\xi}}{\sqrt{1 + |\boldsymbol{\xi}|^2}}\Big) \ .$$

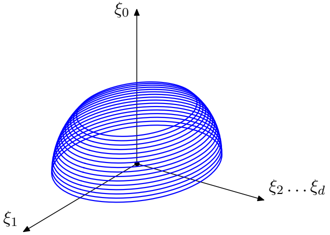
$\mathcal{J} := \mathcal{R} \circ \mathcal{T} : \mathbf{R}^d_* \to \mathrm{S}^d_{(0,r_1)}$

$$\mathcal{R}(\mathbf{R}^d \setminus \mathrm{K}[\mathbf{0}, r_0]) = \left\{ (\zeta_0, \boldsymbol{\zeta}) \in \mathrm{S}^d : 0 < \zeta_0 < r_1 \right\} =: \mathrm{S}^d_{(0,r_1)}$$

and

$$\mathcal{J}(\boldsymbol{\xi}) = \left(\frac{1}{\sqrt{1 + (|\boldsymbol{\xi}| + r_0)^2}}, \frac{|\boldsymbol{\xi}| + r_0}{\sqrt{1 + (|\boldsymbol{\xi}| + r_0)^2}} \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right).$$





Fourier multipliers

Functions from $\mathrm{C}^{\kappa}(\mathrm{S}^{d-1})$, as well as those from $\mathcal{S}(\mathbf{R}^d)$ can be identified as functions on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

Fourier multipliers

Functions from $\mathrm{C}^\kappa(\mathrm{S}^{d-1})$, as well as those from $\mathcal{S}(\mathbf{R}^d)$ can be identified as functions on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

Theorem. *Any function from* $\mathrm{C}^{\lfloor \frac{d}{2} \rfloor+1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ *satisfies Mihlin's condition*

$$|\partial^{\boldsymbol{\alpha}}\psi(\boldsymbol{\xi})| \leqslant \frac{C}{|\boldsymbol{\xi}|^{|\boldsymbol{\alpha}|}}\,, \quad \boldsymbol{\xi} \in \mathbf{R}^d_*\,,$$

for each $|\boldsymbol{\alpha}| \leqslant \lfloor \frac{d}{2} \rfloor + 1$*, when suitably restricted to* $\mathbf{R}^d_*$.
In particular, for any $p \in (1,\infty)$*, it holds (*$\mathcal{A}_\psi \mathsf{u} := (\psi\hat{\mathsf{u}})^\vee$*)*

$$\|\mathcal{A}_\psi\|_{\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))} \leqslant C_{d,p} C_d \|\psi\|_{\mathrm{C}^{\lfloor \frac{d}{2} \rfloor+1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))}\,,$$

for $\psi \in \mathrm{C}^{\lfloor \frac{d}{2} \rfloor+1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$*, where* $C_{d,p}$ *is the constant from the Mihlin theorem, while* C_d *is a constant depending only on* d. ∎

First commutation lemma

Lema. *Let* $\psi \in \mathrm{C}^{\lfloor \frac{d}{2} \rfloor + 1}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$, $\varphi \in \mathrm{C}_0(\mathbf{R}^d)$, $\omega_n \to 0^+$, *and denote* $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$. *Then the commutator of multiplication* B_φ *by* φ *and the Fourier multiplier* $\mathcal{A}_{\psi_n}$ *can be expressed as a sum*

$$C_n := [B_\varphi, \mathcal{A}_{\psi_n}] = \tilde{C}_n + K \,,$$

where for any $p \in (1, \infty)$ *we have that* K *is a compact operator on* $\mathrm{L}^p(\mathbf{R}^d)$, *while* $\tilde{C}_n \to 0$ *in the operator norm on* $\mathcal{L}(\mathrm{L}^p(\mathbf{R}^d))$. ∎

Anisotropic distributions

Let $\Omega \subseteq \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^r_{\mathbf{y}}$, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$.
Spaces of test functions:

$$\mathrm{C}^{l,m}(\Omega)\,, \quad \mathrm{C}^{l,m}_K(\Omega) \quad \text{and} \quad \mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}^{l,m}_{K_n}(\Omega)\,,$$

Anisotropic distributions

Let $\Omega \subseteq \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^r_{\mathbf{y}}$, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$.
Spaces of test functions:

$$\mathrm{C}^{l,m}(\Omega)\,, \quad \mathrm{C}^{l,m}_K(\Omega) \quad \text{and} \quad \mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}^{l,m}_{K_n}(\Omega)\,,$$

and distributions $\quad \mathcal{D}'_{l,m}(\Omega) := (\mathrm{C}^{l,m}_c(\Omega))'$.

Anisotropic distributions

Let $\Omega \subseteq \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^r_{\mathbf{y}}$, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$.
Spaces of test functions:

$$\mathrm{C}^{l,m}(\Omega)\;, \quad \mathrm{C}^{l,m}_K(\Omega) \quad \text{and} \quad \mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}^{l,m}_{K_n}(\Omega)\;,$$

and distributions $\quad \mathcal{D}'_{l,m}(\Omega) := (\mathrm{C}^{l,m}_c(\Omega))'$.
The definition can easily be extended to *differential manifolds without boundary of dimension* d.

Anisotropic distributions

Let $\Omega \subseteq \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^r_{\mathbf{y}}$, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$.
Spaces of test functions:

$$\mathrm{C}^{l,m}(\Omega)\,, \quad \mathrm{C}^{l,m}_K(\Omega) \quad \text{and} \quad \mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n\in\mathbf{N}} \mathrm{C}^{l,m}_{K_n}(\Omega)\,,$$

and distributions $\quad \mathcal{D}'_{l,m}(\Omega) := (\mathrm{C}^{l,m}_c(\Omega))'$.
The definition can easily be extended to *differential manifolds without boundary of dimension* d.

Teorem. *Let X and Y be differential manifolds, of dimension d and r, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$. Then the following statements hold:*

i) *If $K \in \mathcal{D}'_{l,m}(X \times Y)$, then for each $\varphi \in \mathrm{C}^l_c(X)$ the linear form K_φ, defined by $\psi \mapsto \langle K, \varphi \otimes \psi\rangle$, is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ with its strict inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.*

ii) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator, in the pair of topologies as in (i) above. Then there exists a unique distribution of anisotropic order $K \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$ such that for any $\varphi \in \mathrm{C}^l_c(X)$ and $\psi \in \mathrm{C}^{r(m+2)}_c(Y)$ one has*

$$\langle K, \varphi \otimes \psi\rangle = \langle K_\varphi, \psi\rangle = \langle A\varphi, \psi\rangle\,.$$

∎

to manifolds with boundary

Test functions

Let $\Omega \subseteq \mathbb{R}^d_{\mathbf{x}} \times \mathbb{R}^r_{\mathbf{y}}$, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$.

$$\mathrm{C}^{l,m}(\Omega) := \Big\{ f \in \mathrm{C}(\Omega) : (\forall \boldsymbol{\alpha} \in \mathbf{N}_0^d)(\forall \boldsymbol{\beta} \in \mathbf{N}_0^r) \\ |\boldsymbol{\alpha}| \leqslant l \,\&\, |\boldsymbol{\beta}| \leqslant m \implies \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} \partial_{\mathbf{y}}^{\boldsymbol{\beta}} f \in \mathrm{C}(\Omega) \Big\} \,,$$

In a standard way introduce the seminorms using a nested sequence of compacts K_n.

$$\mathrm{C}^{l,m}_K(\Omega) := \Big\{ f \in \mathrm{C}^{l,m}(\Omega) : \ \mathrm{supp}\, f \subseteq K \Big\}$$

is a Banach space for finite l, m, and a Fréchet space for at least one of them infinite.

$$\mathrm{C}^{l,m}_c(\Omega) := \bigcup_{n \in \mathbf{N}} \mathrm{C}^{l,m}_{K_n}(\Omega)$$

with the topology of strict inductive limit is a complete locally convex topological vector space.

Anisotropic distributions

The space of anisotropic distributions is the dual of $\mathrm{C}_c^{l,m}(\Omega)$

$$\mathcal{D}'_{l,m}(\Omega) := (\mathrm{C}_c^{l,m}(\Omega))' \,.$$

In fact

$$T \in \mathcal{D}'_{l,m}(\Omega) \iff \begin{cases} T \in \mathcal{D}'(\Omega)\,, \text{ and} \\ (\forall K \subset\subset \Omega)(\exists C > 0)(\forall \varphi \in \mathrm{C}_K^\infty(\Omega)) \quad |\langle T, \varphi\rangle| \leqslant C p_K^{l,m}(\varphi)\,, \end{cases}$$

The definition can easily be extended to *differential manifolds without boundary of dimension* d:
a locally Euclidean (of the fixed dimension d, i.e. locally diffeomorphic to $\mathbf{R}^d$) second countable Hausdorff topological space on which an equivalence class of C^∞ smooth atlases is given.

Kernel theorem on manifolds without boundary

Teorem. *Let X and Y be differential manifolds, of dimension d and r, and $l, m \in \mathbf{N}_0 \cup \{\infty\}$. Then the following statements hold:*

i) *If $K \in \mathcal{D}'_{l,m}(X \times Y)$, then for each $\varphi \in \mathrm{C}^l_c(X)$ the linear form K_φ, defined by $\psi \mapsto \langle K, \varphi \otimes \psi \rangle$, is a distribution of order not more than m on Y. Furthermore, the mapping $\varphi \mapsto K_\varphi$, taking $\mathrm{C}^l_c(X)$ with its strict inductive limit topology to $\mathcal{D}'_m(Y)$ with weak $*$ topology, is linear and continuous.*

ii) *Let $A : \mathrm{C}^l_c(X) \to \mathcal{D}'_m(Y)$ be a continuous linear operator, in the pair of topologies as in (i) above. Then there exists a unique distribution of anisotropic order $K \in \mathcal{D}'_{l,r(m+2)}(X \times Y)$ such that for any $\varphi \in \mathrm{C}^l_c(X)$ and $\psi \in \mathrm{C}^{r(m+2)}_c(Y)$ one has*

$$\langle K, \varphi \otimes \psi \rangle = \langle K_\varphi, \psi \rangle = \langle A\varphi, \psi \rangle \ .$$

∎

Anisotropic distributions on manifolds with boundary

The definition of *differential manifold with boundary* differs from the notion of a differential manifold without boundary only in that the former is diffeomorphic either to $\mathbf{R}^d$ or to the closed half-space $\operatorname{Cl}\mathbf{R}^d_+ = \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbf{R}^d : x_d \geqslant 0\}$.

Anisotropic distributions on manifolds with boundary

The definition of *differential manifold with boundary* differs from the notion of a differential manifold without boundary only in that the former is diffeomorphic either to $\mathbf{R}^d$ or to the closed half-space $\operatorname{Cl}\mathbf{R}^d_+ = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d : x_d \geqslant 0\}$.

For simplicity, we shall consider only $X = \Omega \subseteq \mathbf{R}^d$ open, and $Y = \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

Anisotropic distributions on manifolds with boundary

The definition of *differential manifold with boundary* differs from the notion of a differential manifold without boundary only in that the former is diffeomorphic either to $\mathbf{R}^d$ or to the closed half-space $\mathrm{Cl}\,\mathbf{R}^d_+ = \{\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d : x_d \geqslant 0\}$.

For simplicity, we shall consider only $X = \Omega \subseteq \mathbf{R}^d$ open, and $Y = \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

The space of distributions on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ of order $l \in \mathbf{N} \cup \{\infty\}$ we define by

$$\mathcal{D}'_l(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) = \left(\mathrm{C}^l(\mathrm{K}_{0,\infty}(\mathbf{R}^d))\right)',$$

where the case $l = \infty$ we shall also denote by $\mathcal{D}'(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
[This corresponds to supported distributions of R. Melrose.]

Anisotropic distributions on manifolds with boundary

The definition of *differential manifold with boundary* differs from the notion of a differential manifold without boundary only in that the former is diffeomorphic either to $\mathbf{R}^d$ or to the closed half-space Cl $\mathbf{R}^d_+ = \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbf{R}^d : x_d \geqslant 0\}$.

For simplicity, we shall consider only $X = \Omega \subseteq \mathbf{R}^d$ open, and $Y = \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

The space of distributions on $\mathrm{K}_{0,\infty}(\mathbf{R}^d)$ of order $l \in \mathbf{N} \cup \{\infty\}$ we define by

$$\mathcal{D}'_l(\mathrm{K}_{0,\infty}(\mathbf{R}^d)) = \left(\mathrm{C}^l(\mathrm{K}_{0,\infty}(\mathbf{R}^d))\right)',$$

where the case $l = \infty$ we shall also denote by $\mathcal{D}'(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$.
[This corresponds to supported distributions of R. Melrose.]

The space of anisotropic distributions on $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$ of order $(l, m) \in (\mathbf{N} \cup \{\infty\})^2$ is defined by

$$\mathcal{D}'_{l,m}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)) = \left(\mathrm{C}^{l,m}_c(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))\right)'.$$

Kernel theorem on $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$

Note that it is sufficient to introduce distributions on $\Omega \times \mathrm{S}^d_{[0,r_1]}$ since by applying the pushforward $(\mathcal{J}^{-1})_*$ we have a one-to-one correspondence with distributions on $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

Kernel theorem on $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$

Note that it is sufficient to introduce distributions on $\Omega \times \mathrm{S}^d_{[0,r_1]}$ since by applying the pushforward $(\mathcal{J}^{-1})_*$ we have a one-to-one correspondence with distributions on $\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

Corollary. *Let $\Omega \subseteq \mathbf{R}^d$ be open and $l, m \in \mathbf{N}_0 \cup \{\infty\}$. Furthermore, let $A : \mathrm{C}^l_c(\Omega) \to \mathcal{D}'_m(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ be a continuous linear operator, taking $\mathrm{C}^l_c(\Omega)$ with its inductive limit topology and $\mathcal{D}'_m(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ with weak $*$ topology. Then there exists a unique distribution of anisotropic order $K \in \mathcal{D}'_{l,d(m+2)}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that for any $\varphi \in \mathrm{C}^l_c(X)$ and $\psi \in \mathrm{C}^{d(m+2)}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$ one has*

$$\langle K, \varphi \otimes \psi \rangle = \langle A\varphi, \psi \rangle \ .$$

∎

$\Omega \subseteq \mathbf{R}^d$ open

Teorem

If $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ and $\omega_n \to 0^+$, then there exist $(u_{n'})$, $(v_{n'})$ and $\mu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 v_{n'}}(\boldsymbol{\xi})} \psi(\omega_{n'}\boldsymbol{\xi})\, d\boldsymbol{\xi} = \langle \mu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \ .$$

The measure $\mu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})}$ is called the one-scale H-measure with characteristic length $(\omega_{n'})$ associated to the (sub)sequences $(u_{n'})$ and $(v_{n'})$.

$\Omega \subseteq \mathbf{R}^d$ open

Teorem

If $u_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ and $\omega_n \to 0^+$, then there exist $(u_{n'})$, $(v_{n'})$ and $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{M}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ and $\psi \in \mathrm{C}(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_n}(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} = \langle \mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1\bar{\varphi}_2 \boxtimes \psi \rangle \ .$$

The measure $\mu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ is called the one-scale H-measure with characteristic length $(\omega_{n'})$ associated to the (sub)sequences $(u_{n'})$ and $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi\hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n\boldsymbol{\xi})$

One-scale H-distributions

$\Omega \subseteq \mathbf{R}^d$ open

Teorem

If $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ in $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega)$ and $\omega_n \to 0^+$, then there exist $(u_{n'})$, $(v_{n'})$ and $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\Omega)$ and $\psi \in E$

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_n}(\varphi_1 u_{n'})(\mathbf{x})\overline{(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} = \langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \ .$$

The distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ is called the one-scale H-distribution with characteristic length $(\omega_{n'})$ associated to the (sub)sequences $(u_{n'})$ and $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

$\Omega \subseteq \mathbf{R}^d$ open, $p \in \langle 1, \infty \rangle$, $\frac{1}{p} + \frac{1}{p'} = 1$

Teorem

If $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, $v_n \rightharpoonup 0$ in $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega)$ and $\omega_n \to 0^+$, then there exist $(u_{n'})$, $(v_{n'})$ and $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}} \in \mathcal{D}'(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$ such that for any $\varphi_1, \varphi_2 \in \mathrm{C}^\infty_c(\Omega)$ and $\psi \in E$

$$\lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_n}(\varphi_1 u_{n'})(\mathbf{x}) \overline{(\varphi_2 v_{n'})(\mathbf{x})} \, d\mathbf{x} = \langle \nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}, \varphi_1 \bar{\varphi}_2 \boxtimes \psi \rangle \ .$$

The distribution $\nu^{(\omega_{n'})}_{\mathrm{K}_{0,\infty}}$ is called the one-scale H-distribution with characteristic length $(\omega_{n'})$ associated to the (sub)sequences $(u_{n'})$ and $(v_{n'})$.

$\mathcal{A}_\psi(u) = (\psi \hat{u})^\vee$, $\psi_n(\boldsymbol{\xi}) := \psi(\omega_n \boldsymbol{\xi})$

Determine E such that

— $\mathcal{A}_\psi : \mathrm{L}^p(\mathbf{R}^d) \longrightarrow \mathrm{L}^p(\mathbf{R}^d)$ is continuous

— The First commutation lemma is valid

Existence of one-scale H-distributions

Teorem. *Let* $\Omega \subseteq \mathbf{R}^d$ *be open. If* $u_n \rightharpoonup 0$ *in* $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$ *and* (v_n) *is bounded in* $\mathrm{L}^q_{\mathrm{loc}}(\Omega)$ *(for some* $p \in (1, \infty)$ *and* $q \geqslant p'$*, where* $1/p + 1/p' = 1$*), and if* $\omega_n \to 0^+$*, then there exist subsequences* $(u_{n'})$*,* $(v_{n'})$*, and a complex valued (supported) distribution* $\nu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})} \in \mathcal{D}'_{0,\kappa}(\Omega \times \mathrm{K}_{0,\infty}(\mathbf{R}^d))$*, where* $\kappa := d(\lfloor \frac{d}{2} \rfloor + 3)$*, such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\Omega)$ *and* $\psi \in \mathrm{C}^\kappa(\mathrm{K}_{0,\infty}(\mathbf{R}^d))$*, one has:*

$$\begin{aligned}\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{A}_{\psi_{n'}}(\varphi_1 u_{n'})(\mathbf{x}) \overline{(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} &= \lim_{n'\to\infty} \int_{\mathbf{R}^d} (\varphi_1 u_{n'})(\mathbf{x}) \overline{\mathcal{A}_{\bar{\psi}_{n'}}(\varphi_2 v_{n'})(\mathbf{x})}\, d\mathbf{x} \\ &= \left\langle \nu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})}, \varphi_1 \bar{\varphi}_2 \otimes \psi \right\rangle,\end{aligned} \tag{1}$$

where $\psi_n = \psi(\omega_n \cdot)$*. The distribution* $\nu_{\mathrm{K}_{0,\infty}}^{(\omega_{n'})}$ *we call the* one-scale H-distribution (with the characteristic length $(\omega_{n'})$) *associated to (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$*.*

Moreover, for $p = 2$ *the one-scale H-distribution above is the one-scale H-measures with characteristic length* $(\omega_{n'})$ *associated to (sub)sequences* $(u_{n'})$ *and* $(v_{n'})$*.*

∎

Immediate properties of one-scale H-distributions

Changing the order of sequences; (v_n) and (u_n) determine the distribution

$$\left\langle \bar{\nu}_{\mathrm{K}_{0,\infty}}, \Psi \right\rangle = \overline{\left\langle \nu_{\mathrm{K}_{0,\infty}}, \bar{\Psi} \right\rangle} .$$

Immediate properties of one-scale H-distributions

Changing the order of sequences; (v_n) and (u_n) determine the distribution

$$\left\langle \bar{\nu}_{\mathrm{K}_{0,\infty}}, \Psi \right\rangle = \overline{\left\langle \nu_{\mathrm{K}_{0,\infty}}, \bar{\Psi} \right\rangle}\,.$$

Supports: if u_n, v_n are supported in closed sets $F_1, F_2 \subseteq \Omega$, then any one-scale distribution they determine is supported in $(F_1 \cap F_2) \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

Immediate properties of one-scale H-distributions

Changing the order of sequences; (v_n) and (u_n) determine the distribution

$$\left\langle \bar{\nu}_{\mathrm{K}_{0,\infty}}, \Psi \right\rangle = \overline{\left\langle \nu_{\mathrm{K}_{0,\infty}}, \bar{\Psi} \right\rangle}\,.$$

Supports: if u_n, v_n are supported in closed sets $F_1, F_2 \subseteq \Omega$, then any one-scale distribution they determine is supported in $(F_1 \cap F_2) \times \mathrm{K}_{0,\infty}(\mathbf{R}^d)$.

Lema. *Let $u_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$, for some $p \in \langle 1, \infty \rangle$. Then the following statements are equivalent:*

(a) $u_n \to 0$ *(strongly) in* $\mathrm{L}^p_{\mathrm{loc}}(\Omega)$.

(b) *For every bounded sequence (v_n) in $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega)$ and every $\omega_n \to 0^+$, (u_n) and (v_n) form an (ω_n)-pure pair and the corresponding one-scale H-distribution is zero.*

(c) *For $v_n = |u_n|^{p-2}u_n$ and some $\omega_n \to 0^+$, (u_n) and (v_n) form an (ω_n)-pure pair and the corresponding one-scale H-distribution is zero.*

∎

Localisation principle for one-scale H-distributions ...

Theorem. Let $\mathsf{u}_n \rightharpoonup 0$ in $\mathrm{L}^p_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ satisfy

$$\sum_{|\boldsymbol{\alpha}| \leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|} \partial_{\boldsymbol{\alpha}} (\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) = \mathsf{f}_n \,,$$

where (ε_n) is a sequence of positive real numbers, $\mathbf{A}_n^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$, such that for any $\boldsymbol{\alpha} \in \mathbf{N}_0^d$ the sequence $\mathbf{A}_n^{\boldsymbol{\alpha}} \to \mathbf{A}^{\boldsymbol{\alpha}}$ in the space $\mathrm{C}(\Omega; \mathrm{M}_{\mathrm{q}\times\mathrm{r}}(\mathbf{C}))$ (in other words, $\mathbf{A}_n^{\boldsymbol{\alpha}}$ converges locally uniformly to $\mathbf{A}^{\boldsymbol{\alpha}}$), while (f_n) is a sequence of functions in $\mathrm{W}^{-m,p}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ satisfying (ε_n)-local compactness condition

$$(\forall \varphi \in \mathrm{C}_c^\infty(\Omega)) \qquad \mathcal{A}_{\frac{1}{1+|\varepsilon_n \boldsymbol{\xi}|^m}} (\varphi \mathsf{f}_n) \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^r)\,.$$

Moreover, let (v_n) be a bounded sequence in $\mathrm{L}^{p'}_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$ and let $\omega_n \to 0^+$ be a sequence of positive reals such that $c := \lim_n \frac{\omega_n}{\varepsilon_n}$ exists (in $[0, \infty]$).

Localisation principle for one-scale H-distributions (cont.)

Then any one-scale H-distribution $\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}}^{(\omega_n)}$ associated to (sub)sequences (of) (u_n) and (v_n) with characteristic length (ω_n) satisfies:

$$\mathbf{p}_c(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\nu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}\,,$$

where, with respect to the value of c, we have

i) $c = 0$:

$$\mathbf{p}_0(\mathbf{x},\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}|=m} (2\pi i)^m \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{1+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,,$$

ii) $c \in (0,\infty)$:

$$\mathbf{p}_c(\mathbf{x},\boldsymbol{\xi}) = \sum_{|\boldsymbol{\alpha}|\leqslant m} \left(\frac{2\pi i}{c}\right)^{|\boldsymbol{\alpha}|} \frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{1+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,,$$

iii) $c = \infty$:

$$\mathbf{p}_\infty(\mathbf{x},\boldsymbol{\xi}) = \frac{1}{1+|\boldsymbol{\xi}|^m}\mathbf{A}^{0}(\mathbf{x})\,.$$

to localisation for H-measures

Localisation principle for one-scale H-measures ($c = 1$)

$$\sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} \varepsilon_n^{|\boldsymbol{\alpha}|-l}\partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}}\mathsf{u}_n) = \mathsf{f}_n \quad \text{in } \Omega\,,$$

$$(\forall\,\varphi\in \mathrm{C}_c^\infty(\Omega)) \qquad \frac{\widehat{\varphi\mathsf{f}_n}}{1+\sum_{s=l}^m \varepsilon_n^{s-l}|\boldsymbol{\xi}|^s} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)\,.$$

Theorem. *Under previous assumptions, one-scale H-measure* $\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}}$ *with characteristic length* (ε_n) *corresponding to* (u_n) *satisfies*

$$\mathbf{p}\boldsymbol{\mu}_{\mathrm{K}_{0,\infty}} = \mathbf{0}\,,$$

where

$$\mathbf{p}(\mathbf{x},\boldsymbol{\xi}) := \sum_{l\leqslant|\boldsymbol{\alpha}|\leqslant m} (2\pi i)^{|\boldsymbol{\alpha}|}\frac{\boldsymbol{\xi}^{\boldsymbol{\alpha}}}{|\boldsymbol{\xi}|^l+|\boldsymbol{\xi}|^m}\mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

Localisation principle for H-measures

Theorem. *If* $\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there is a subsequence* $(\mathsf{u}^{n'})$ *and* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that:*

$$\lim_{n'\to\infty} \int\limits_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}^{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}^{n'}\Big) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \psi \rangle$$

$$= \int\limits_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi}) \, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}) \; .$$

∎

$$\sum_k \partial_k \Big(\mathbf{A}^k \mathsf{u}^n\Big) + \mathbf{B}\mathsf{u}^n = \mathsf{f}^n \;, \; \mathbf{A}^k \text{ Hermitian}, \quad \mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

Localisation principle for H-measures

Theorem. *If* $\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there is a subsequence* $(\mathsf{u}^{n'})$ *and* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}^{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}^{n'}\Big) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi}) \, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}) \ .$$

∎

$$\sum_k \partial_k \Big(\mathbf{A}^k \mathsf{u}^n\Big) + \mathbf{B}\mathsf{u}^n = \mathsf{f}^n \ , \ \mathbf{A}^k \text{ Hermitian}, \quad \mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

If supports of $\mathsf{u}^n, \mathsf{f}^n$ are contained inside Ω, we can extend them by zero to $\mathbf{R}^d$.

Theorem. (localisation property) *If* $\mathsf{u}^n \rightharpoonup 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)^r$ *defines* $\boldsymbol{\mu}$, *and if* u^n *satisfies:*

$$\sum_k \partial_k (\mathbf{A}^k \mathsf{u}^n) \to 0 \ \text{ in the space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \sum_k \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\Omega \times S^{d-1}$ *it holds:* $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) \boldsymbol{\mu} = \mathbf{0}$.

∎

Localisation principle for H-measures

Theorem. *If* $\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there is a subsequence* $(\mathsf{u}^{n'})$ *and* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}^{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}^{n'}\Big) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi}) \, d\boldsymbol{\mu}(\mathbf{x}, \boldsymbol{\xi}) \; .$$

∎

$$\sum_k \partial_k \Big(\mathbf{A}^k \mathsf{u}^n\Big) + \mathbf{B}\mathsf{u}^n = \mathsf{f}^n \;\; , \; \mathbf{A}^k \text{ Hermitian}, \quad \mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

If supports of $\mathsf{u}^n, \mathsf{f}^n$ are contained inside Ω, we can extend them by zero to $\mathbf{R}^d$.

Theorem. (localisation property) *If* $\mathsf{u}^n \rightharpoonup 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)^r$ *defines* $\boldsymbol{\mu}$*, and if* u^n *satisfies:*

$$\sum_k \partial_k (\mathbf{A}^k \mathsf{u}^n) \to 0 \;\; \text{in the space} \, \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \; ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \sum_k \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\Omega \times S^{d-1}$ *it holds:* $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu} = \mathbf{0}$.

∎

Thus, the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\big\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\big\}$ of points where $\mathbf{P}$ is a singular matrix.

Localisation principle for H-measures

Theorem. *Let* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{C}^r)$, *and let for a given* $m \in \mathbf{N}$

$$\sum_{|\boldsymbol{\alpha}| \leqslant m} \partial_{\boldsymbol{\alpha}}(\mathbf{A}^{\boldsymbol{\alpha}} \mathsf{u}_n) \longrightarrow 0 \quad \textit{strongly in} \quad \mathrm{H}^{-m}_{\mathrm{loc}}(\Omega; \mathrm{C}^q)\,,$$

where $\mathbf{A}^{\boldsymbol{\alpha}} \in \mathrm{C}(\Omega; \mathrm{M}_{q\times r}(\mathbf{C}))$ *and* $\partial_{\boldsymbol{\alpha}} = \frac{\partial^{\alpha_1}}{\partial x^1} \dots \frac{\partial^{\alpha_d}}{\partial x^d}$ *denotes partial derivatives in variable* $\mathbf{x}$ *in the physical space.*
Then for the associated H-measure $\boldsymbol{\mu}$ *we have*

$$\mathbf{p}_{pr} \boldsymbol{\mu} = \mathbf{0}\,,$$

where the principal symbol of the differential operator is

$$\mathbf{p}_{pr}(\mathbf{x}, \boldsymbol{\xi}) := \sum_{|\boldsymbol{\alpha}| = m} (2\pi i)^m \Big(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\Big)^{\boldsymbol{\alpha}} \mathbf{A}^{\boldsymbol{\alpha}}(\mathbf{x})\,.$$

∎

This result implies that the support of $\boldsymbol{\mu}$ is contained in the set

$$\Sigma_{\mathbf{p}_{pr}} := \Big\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times \mathrm{S}^{d-1} : \operatorname{rank} \mathbf{p}_{pr}(\mathbf{x}, \boldsymbol{\xi}) < r\Big\}$$

of points where $\mathbf{p}_{pr}(\mathbf{x}, \boldsymbol{\xi})$ is not left invertible.

The wave equation

Assumptions for Kirchhoff-Love plates

- the plate is thin, but not very thin
  (rougly, the thickness is 1–20% of the leading dimension)
- the plate thickness might vary only slowly
  (so that the 3D stress effects are ignored)
- the plate is symmetric about mid-surface
- applied transverse loads are distributed over plate surface areas (no concentrated loads)
- there is no significant extension of the mid-surface
  There are no transverse shear deformations.
  The variation of vertical displacement in the direction of thickness can be neglected.
  The planes perpendicular to the mid-surface will remain plane and perpendicular to the deformed mid-surface.

Kirchhoff-Love plate equation

The above leads to a linear elliptic problem, with homogeneous Dirichlet boundary conditions:

$$\left\{\begin{array}{l}\operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}_0^2(\Omega)\,,\end{array}\right.$$

Kirchhoff-Love plate equation

The above leads to a linear elliptic problem, with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

where:

- $\Omega \subseteq \mathbf{R}^d$ is a bounded domain ($d = 2$ ... for the plate)
- $f \in \mathrm{H}^{-2}(\Omega)$ is the external load
- $u \in \mathrm{H}^2_0(\Omega)$ is the vertical displacement of the plate

Kirchhoff-Love plate equation

The above leads to a linear elliptic problem, with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}_0^2(\Omega)\,, \end{cases}$$

where:

- $\Omega \subseteq \mathbf{R}^d$ is a bounded domain ($d = 2$ … for the plate)
- $f \in \mathrm{H}^{-2}(\Omega)$ is the external load
- $u \in \mathrm{H}_0^2(\Omega)$ is the vertical displacement of the plate
- **M** describes (non-homogeneous) properties of the material plate is made of. At a point it is a linear operator from symmetric matrices to symmetric matrices, and we take **M** from the set:

$$\mathfrak{M}_2(\alpha,\beta;\Omega) := \Big\{ \mathbf{N} \in \mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym})$$

$$\mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geqslant \alpha \mathbf{S} : \mathbf{S} \ (\text{ae } \mathbf{x}) \ \& \ \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geqslant \frac{1}{\beta}\mathbf{S} : \mathbf{S} \ (\text{ae } \mathbf{x}) \Big\}$$

This ensures the boundedness and coercivity, so we have the existence and uniqueness of solutions via the Lax-Milgram lemma in a standard way.

Homogenisation: H-convergence

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in \mathrm{H}^{-2}(\Omega)$ the sequence of solutions u_n of problems

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

coverges weakly to a limit u in $\mathrm{H}^2_0(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $\mathrm{L}^2(\Omega; \mathrm{Sym})$.

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha,\beta;\Omega)$ H-converges to $\mathbf{M}\in\mathfrak{M}_2(\alpha',\beta';\Omega)$ if for any $f\in \mathrm{H}^{-2}(\Omega)$ the sequence of solutions u_n of problems

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n\nabla\nabla u_n)=f \quad \text{in} \quad \Omega \\ u_n\in \mathrm{H}^2_0(\Omega) \end{cases}$$

coverges weakly to a limit u in $\mathrm{H}^2_0(\Omega)$, while the sequence $(\mathbf{M}^n\nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $\mathrm{L}^2(\Omega;\mathrm{Sym})$.

This convergence comes indeed from a weak topology on $X=\bigcup\mathfrak{M}_2(1/n,n;\Omega)$, where we consider the maps $\mathbf{M}\mapsto u$, with weak topology on $\mathrm{H}^2_0(\Omega)$, for any fixed $f\in\mathrm{H}^{-2}(\Omega)$, as well as $\mathbf{M}\mapsto\mathbf{M}\nabla\nabla u$, with weak topology on $\mathrm{L}^2(\Omega;\mathrm{Sym})$.

Homogenisation: H-convergence

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in \mathrm{H}^{-2}(\Omega)$ the sequence of solutions u_n of problems

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

coverges weakly to a limit u in $\mathrm{H}^2_0(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $\mathrm{L}^2(\Omega; \mathrm{Sym})$.

This convergence comes indeed from a weak topology on $X = \bigcup \mathfrak{M}_2(1/n, n; \Omega)$, where we consider the maps $\mathbf{M} \mapsto u$, with weak topology on $\mathrm{H}^2_0(\Omega)$, for any fixed $f \in \mathrm{H}^{-2}(\Omega)$, as well as $\mathbf{M} \mapsto \mathbf{M}\nabla\nabla u$, with weak topology on $\mathrm{L}^2(\Omega; \mathrm{Sym})$.

for second order elliptic equations:
Tartar & Murat, 1977

general form for higher-order elliptic equations:
Žikov, Kozlov, Oleinik, Ngoan, 1979

for plates: N.A. & N. Balenović, 1999–2000
revisited: K. Burazin, J. Jankov (& M. Vrdoljak), 2018–21

Properties: Compactness

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$. *Then there is a subsequence* $(\mathbf{M}^{n_k})$ *and a tensor function* $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *such that* $(\mathbf{M}^{n_k})$ *H-converges to* $\mathbf{M}$. ∎

Properties: Compactness

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$. *Then there is a subsequence* $(\mathbf{M}^{n_k})$ *and a tensor function* $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *such that* $(\mathbf{M}^{n_k})$ *H-converges to* $\mathbf{M}$. ∎

Theorem. *(compactness by compensation) Let the following convergences be valid:*

$$
\begin{aligned}
w^n &\rightharpoonup w^\infty \quad \text{in} \quad \mathrm{H}^2_{\mathrm{loc}}(\Omega)\,,\\
\mathbf{D}^n &\rightharpoonup \mathbf{D}^\infty \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})\,,
\end{aligned}
$$

with an additional assumption that the sequence $(\operatorname{div}\operatorname{div}\mathbf{D}^n)$ *is contained in a precompact (for the strong topology) set of the space* $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$. *Then we have*

$$
\nabla\nabla w^n : \mathbf{D}^n \overset{*}{\rightharpoonup} \nabla\nabla w^\infty : \mathbf{D}^\infty
$$

in the space of Radon measures. ∎

to Dependence on parameters

Locality and irrelevance of boundary conditions

Theorem. *(locality of H-convergence) Let* $(\mathbf{M}^n)$ *and* $(\mathbf{O}^n)$ *be two sequences of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, which H-converge to* $\mathbf{M}$ *and* $\mathbf{O}$*, respectively. Let* ω *be an open subset compactly embedded in* Ω*. If* $\mathbf{M}^n(\mathbf{x}) = \mathbf{O}^n(\mathbf{x})$ *in* ω*, then* $\mathbf{M}(\mathbf{x}) = \mathbf{O}(\mathbf{x})$ *in* ω*.* ∎

Theorem. *(irrelevance of boundary conditions) Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converges to* $\mathbf{M}$*. For any sequence* (z_n) *such that*

$$\begin{aligned} z_n &\longrightarrow z && \text{in } \mathrm{H}^2_{\mathrm{loc}}(\Omega) \\ \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla z_n) = f_n &\longrightarrow f && \text{in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega), \end{aligned}$$

the weak convergence $\mathbf{M}^n \nabla\nabla z_n \rightharpoonup \mathbf{M}\nabla\nabla z$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$ *holds.* ∎

Convergence of energies

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha,\beta;\Omega)$ *that H-converges to* $\mathbf{M}$. *For any* $f \in \mathrm{H}^{-2}(\Omega)$, *the sequence* (u_n) *of solutions of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega) \end{cases}$$

satisfies $\mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \rightharpoonup \mathbf{M}\nabla\nabla u : \nabla\nabla u$ *weakly-*$*$ *in the space of Radon measures and* $\displaystyle\int_\Omega \mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \, d\mathbf{x} \to \int_\Omega \mathbf{M}\nabla\nabla u : \nabla\nabla u \, d\mathbf{x}$, *where* u *is the solution of the homogenised equation*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}_0^2(\Omega)\,. \end{cases}$$

∎

Ordering property for symmetric tensors ...

Theorem. *Let* $(\mathbf{M}^n)$ *and* $(\mathbf{O}^n)$ *be two sequences of symmetric tensors in* $\mathfrak{M}_2(\alpha,\beta;\Omega)$ *that H-converge to the homogenised tensors* $\mathbf{M}$ *and* $\mathbf{O}$*, respectively. Furthermore, assume that, for any* n*,*

$$(\forall\, \boldsymbol{\xi} \in \mathrm{Sym}) \qquad \mathbf{M}^n\boldsymbol{\xi} : \boldsymbol{\xi} \leqslant \mathbf{O}^n\boldsymbol{\xi} : \boldsymbol{\xi}\ .$$

Then the homogenised limits are also ordered:

$$(\forall\, \boldsymbol{\xi} \in \mathrm{Sym}) \qquad \mathbf{M}\boldsymbol{\xi} : \boldsymbol{\xi} \leqslant \mathbf{O}\boldsymbol{\xi} : \boldsymbol{\xi}\ .$$

∎

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha,\beta;\Omega)$ *that either converges strongly to a limit tensor* $\mathbf{M}$ *in* $\mathrm{L}^1(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$*, or converges to* $\mathbf{M}$ *almost everywhere in* Ω*. Then,* $\mathbf{M}^n$ *also H-converges to* $\mathbf{M}$*.* ∎

Theorem. *Let* $F = \{f_n : n \in \mathbf{N}\}$ *be a countable dense family in* $\mathrm{H}^{-2}(\Omega)$, **M** *and* **O** *tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$, *and* (u_n), (v_n) *sequences of solutions to*

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}\nabla\nabla u_n) = f_n \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

and

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{O}\nabla\nabla v_n) = f_n \\ v_n \in \mathrm{H}^2_0(\Omega) \end{cases} .$$

Then,

$$d(\mathbf{M}, \mathbf{O}) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{M}\nabla\nabla u_n - \mathbf{O}\nabla\nabla v_n\|_{\mathrm{H}^{-1}(\Omega;\mathrm{Sym})}}{\|f_n\|_{\mathrm{H}^{-2}(\Omega)}}$$

is a metric on $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *and H-convergence is equivalent to the convergence with respect to* d.

∎

Correctors

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$, and $(w_n^{ij})_{1 \leqslant i,j \leqslant d}$ a family of test functions satisfying

$$\begin{aligned} w_n^{ij} &\rightharpoonup \frac{1}{2} x_i x_j && \text{in} \quad \mathrm{H}^2(\Omega) \\ \mathbf{M}^n \nabla\nabla w_n^{ij} &\rightharpoonup \cdots && \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathsf{Sym}) \\ \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla w_n^{ij}) &\to \cdots && \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega). \end{aligned}$$

The sequence of tensors $\mathbf{W}^n$ defined by $W^n_{ijkm} = [\nabla\nabla w_n^{km}]_{ij}$ is called the sequence of correctors.
It is unique, indeed:

Theorem. *Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor $\mathbf{M}$. A sequence of correctors $(\mathbf{W}^n)$ is unique in the sense that, if there exist two sequences of correctors $(\mathbf{W}^n)$ and $(\tilde{\mathbf{W}}^n)$, their difference $(\mathbf{W}^n - \tilde{\mathbf{W}}^n)$ converges strongly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.* ∎

Corrector result

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *which H-converges to* $\mathbf{M}$*. For* $f \in \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$*, let* (u_n) *be the solution of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

and let u *be the weak limit of* (u_n) *in* $\mathrm{H}^2_0(\Omega)$*, i.e. the solution of the homogenised equation*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M} \nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}^2_0(\Omega)\,. \end{cases}$$

Then $\mathbf{R}_n := \nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u \to \mathbf{0}$ *strongly in* $\mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathrm{Sym})$*.*

∎

Smoothness with respect to a parameter $p \in P$

Theorem. *Let* $\mathbf{M}^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ *be a sequence of tensors, such that* $\mathbf{M}^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$*, for* $p \in P$*. Assume that* $p \mapsto \mathbf{M}^n(\cdot, p)$ *is of class* C^k *from* P *to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, with derivatives (up to order* k*) being equicontinuous on every compact set* $K \subseteq P$*:*

$$(\forall K \in \mathcal{K}(P))(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \leq k)$$
$$|p - q| < \delta \Rightarrow \|(\mathbf{M}^n)^{(i)}(\cdot, p) - (\mathbf{M}^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))} < \varepsilon.$$

Then there is a subsequence $(\mathbf{M}^{n_k})$ *such that for every* $p \in P$

$$\mathbf{M}^{n_k}(\cdot, p) \xrightarrow{H} \mathbf{M}(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega)$$

and $p \mapsto \mathbf{M}(\cdot, p)$ *is a* C^k *mapping from P to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$. ∎

Smoothness with respect to a parameter $p \in P$

Theorem. *Let* $\mathbf{M}^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ *be a sequence of tensors, such that* $\mathbf{M}^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$*, for* $p \in P$*. Assume that* $p \mapsto \mathbf{M}^n(\cdot, p)$ *is of class* C^k *from* P *to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, with derivatives (up to order* k*) being equicontinuous on every compact set* $K \subseteq P$*:*

$$(\forall K \in \mathcal{K}(P))(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \leq k)$$
$$|p - q| < \delta \Rightarrow \|(\mathbf{M}^n)^{(i)}(\cdot, p) - (\mathbf{M}^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))} < \varepsilon.$$

Then there is a subsequence $(\mathbf{M}^{n_k})$ *such that for every* $p \in P$

$$\mathbf{M}^{n_k}(\cdot, p) \xrightarrow{H} \mathbf{M}(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega)$$

and $p \mapsto \mathbf{M}(\cdot, p)$ *is a* C^k *mapping from P to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*.* ∎

In particular, the above is valid for $k = \infty$ and $k = \omega$ (the analytic functions).

Small-amplitude homogenisation

Consider a sequence of problems

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n(\cdot\,;\gamma)\nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

where we assume that the coefficients are a small perturbation of a given continuous tensor function $\mathbf{A}_0$, for small γ

$$\mathbf{M}^n(\cdot\,;\gamma) := \mathbf{A}_0 + \gamma\mathbf{B}^n + \gamma^2\mathbf{C}^n + o(\gamma^2)\;,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{O}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$. For small γ, in fact, we can assume that the function is analytic in γ.
Then (after passing to a subsequence if needed)

$$\mathbf{M}^n(\cdot\,;\gamma) \xrightarrow{H} \mathbf{M}(\cdot\,;\gamma) = \mathbf{A}_0 + \gamma\mathbf{B}_0 + \gamma^2\mathbf{C}_0 + o(\gamma^2)\;;$$

the limit being measurable in $\mathbf{x}$, and analytic in γ.

Small-amplitude homogenisation

Consider a sequence of problems

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n(\cdot\,;\gamma)\nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

where we assume that the coefficients are a small perturbation of a given continuous tensor function $\mathbf{A}_0$, for small γ

$$\mathbf{M}^n(\cdot\,;\gamma) := \mathbf{A}_0 + \gamma\mathbf{B}^n + \gamma^2\mathbf{C}^n + o(\gamma^2)\ ,$$

where $\mathbf{B}^n, \mathbf{C}^n \xrightarrow{*} \mathbf{O}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$. For small γ, in fact, we can assume that the function is analytic in γ.
Then (after passing to a subsequence if needed)

$$\mathbf{M}^n(\cdot\,;\gamma) \xrightarrow{H} \mathbf{M}(\cdot\,;\gamma) = \mathbf{A}_0 + \gamma\mathbf{B}_0 + \gamma^2\mathbf{C}_0 + o(\gamma^2)\ ;$$

the limit being measurable in $\mathbf{x}$, and analytic in γ.

The goal is to obtain the explicit formula for the leading terms $\mathbf{B}_0$ and $\mathbf{C}_0$ in the expansion of the homogenisation limit.

Small-amplitude homogenisation procedure

Take $u \in \mathrm{H}_0^2(\Omega)$ and define $f_\gamma := \operatorname{div}\operatorname{div}(\mathbf{M}(\cdot\,;\gamma)\nabla\nabla u)$, depending analytically on γ.

Small-amplitude homogenisation procedure

Take $u \in \mathrm{H}^2_0(\Omega)$ and define $f_\gamma := \operatorname{div}\operatorname{div}(\mathbf{M}(\cdot\,;\gamma)\nabla\nabla u)$, depending analytically on γ. Using f_γ, let u^n_γ be the solution (for each n and γ) of

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n(\cdot\,;\gamma)\nabla\nabla u^n_\gamma) = f_\gamma \text{ in } \Omega \\ \hfill u^n_\gamma \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

which analytically depends on γ, hence one can write

$$u^n_\gamma := u^n_0 + \gamma u^n_1 + \gamma^2 u^n_2 + o(\gamma^2)\,.$$

Small-amplitude homogenisation procedure

Take $u \in \mathrm{H}^2_0(\Omega)$ and define $f_\gamma := \operatorname{div}\operatorname{div}(\mathbf{M}(\cdot\,;\gamma)\nabla\nabla u)$, depending analytically on γ. Using f_γ, let u^n_γ be the solution (for each n and γ) of

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n(\cdot\,;\gamma)\nabla\nabla u^n_\gamma) = f_\gamma \text{ in } \Omega \\ \qquad\qquad u^n_\gamma \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

which analytically depends on γ, hence one can write

$$u^n_\gamma := u^n_0 + \gamma u^n_1 + \gamma^2 u^n_2 + o(\gamma^2)\,.$$

As $\mathbf{M}^n(\cdot\,;\gamma) \xrightarrow{H} \mathbf{M}(\cdot\,;\gamma)$, we have weak convergences in $\mathrm{L}^2(\Omega;\mathrm{Sym})$:

$$(*) \qquad \begin{aligned} \mathbf{E}^n_\gamma &:= \nabla\nabla u^n_\gamma \rightharpoonup \nabla\nabla u \\ \mathbf{D}^n_\gamma &:= \mathbf{M}^n(\cdot\,;\gamma)\mathbf{E}^n_\gamma \rightharpoonup \mathbf{M}(\cdot\,;\gamma)\nabla\nabla u\,. \end{aligned}$$

$\mathbf{E}^n_\gamma$ and $\mathbf{D}^n_\gamma$ are analytic in γ and consequently each can be expanded in the Taylor series:

$$\begin{aligned} \mathbf{E}^n_\gamma &= \mathbf{E}^n_0 + \gamma\mathbf{E}^n_1 + \gamma^2\mathbf{E}^n_2 + o(\gamma^2) \\ \mathbf{D}^n_\gamma &= \mathbf{D}^n_0 + \gamma\mathbf{D}^n_1 + \gamma^2\mathbf{D}^n_2 + o(\gamma^2)\,. \end{aligned}$$

Small-amplitude homogenisation procedure

Take $u \in \mathrm{H}_0^2(\Omega)$ and define $f_\gamma := \operatorname{div}\operatorname{div}(\mathbf{M}(\cdot\,;\gamma)\nabla\nabla u)$, depending analytically on γ. Using f_γ, let u_γ^n be the solution (for each n and γ) of

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n(\cdot\,;\gamma)\nabla\nabla u_\gamma^n) = f_\gamma \ \text{in } \Omega \\ \qquad\qquad\qquad\qquad u_\gamma^n \in \mathrm{H}_0^2(\Omega)\,, \end{cases}$$

which analytically depends on γ, hence one can write

$$u_\gamma^n := u_0^n + \gamma u_1^n + \gamma^2 u_2^n + o(\gamma^2)\,.$$

As $\mathbf{M}^n(\cdot\,;\gamma) \xrightarrow{H} \mathbf{M}(\cdot\,;\gamma)$, we have weak convergences in $\mathrm{L}^2(\Omega;\mathrm{Sym})$:

$$(*)\qquad \begin{aligned} \mathbf{E}_\gamma^n &:= \nabla\nabla u_\gamma^n \rightharpoonup \nabla\nabla u \\ \mathbf{D}_\gamma^n &:= \mathbf{M}^n(\cdot\,;\gamma)\mathbf{E}_\gamma^n \rightharpoonup \mathbf{M}(\cdot\,;\gamma)\nabla\nabla u\,. \end{aligned}$$

$\mathbf{E}_\gamma^n$ and $\mathbf{D}_\gamma^n$ are analytic in γ and consequently each can be expanded in the Taylor series:

$$\begin{aligned} \mathbf{E}_\gamma^n &= \mathbf{E}_0^n + \gamma\mathbf{E}_1^n + \gamma^2\mathbf{E}_2^n + o(\gamma^2) \\ \mathbf{D}_\gamma^n &= \mathbf{D}_0^n + \gamma\mathbf{D}_1^n + \gamma^2\mathbf{D}_2^n + o(\gamma^2)\,. \end{aligned}$$

For $\gamma = 0$, the uniqueness of solution implies $u_0^n = u$. Moreover, this gives us

$$\mathbf{E}_0^n = \nabla\nabla u \quad \text{and} \quad \mathbf{D}_0^n = \mathbf{A}_0\nabla\nabla u\,.$$

Small-amplitude homogenisation procedure (cont.)

After inserting the above expansions into $(*)$ and equating the terms with equal powers of γ, one can conclude that $\mathbf{E}_1^n, \mathbf{E}_2^n \longrightarrow \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, and

$$\mathbf{D}_1^n = \mathbf{A}_0 \mathbf{E}_1^n + \mathbf{B}^n \nabla\nabla u.$$

Small-amplitude homogenisation procedure (cont.)

After inserting the above expansions into $(*)$ and equating the terms with equal powers of γ, one can conclude that $\mathbf{E}_1^n, \mathbf{E}_2^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, and

$$\mathbf{D}_1^n = \mathsf{A}_0 \mathbf{E}_1^n + \mathsf{B}^n \nabla\nabla u.$$

Since $\mathbf{E}_1^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, while $\mathsf{B}^n \xrightharpoonup{*} \mathsf{O}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$:

$$\mathbf{D}_1^n \rightharpoonup \mathbf{0} \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}) \, .$$

Small-amplitude homogenisation procedure (cont.)

After inserting the above expansions into $(*)$ and equating the terms with equal powers of γ, one can conclude that $\mathbf{E}_1^n, \mathbf{E}_2^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, and

$$\mathbf{D}_1^n = \mathsf{A}_0 \mathbf{E}_1^n + \mathsf{B}^n \nabla\nabla u .$$

Since $\mathbf{E}_1^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$, while $\mathsf{B}^n \overset{*}{\rightharpoonup} \mathsf{O}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$:

$$\mathbf{D}_1^n \rightharpoonup \mathbf{0} \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}) .$$

Similarly, by using

$$\mathbf{D}_\gamma^n = \mathsf{M}^n(\cdot\,;\gamma)\mathbf{E}_\gamma^n \rightharpoonup \mathsf{M}(\cdot\,;\gamma)\nabla\nabla u = (\mathsf{A}_0 + \gamma \mathsf{B}_0 + \gamma^2 \mathsf{C}_0 + o(\gamma^2))\nabla\nabla u \,,$$

after equating the terms standing by γ^1, we obtain that

$$\mathbf{D}_1^n \rightharpoonup \mathsf{B}_0 \nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}) .$$

The limits are equal, so $\mathsf{B}_0 \nabla\nabla u = \mathbf{0}$.
Since $u \in \mathrm{H}_0^2(\Omega)$ can be arbitrary, we conlude that $\mathsf{B}_0 = \mathsf{O}$.

The corrector can be expressed by H-measure

Analogously, equating the terms standing by γ^2 gives:

$$\mathbf{D}_2^n = \mathsf{A}_0\mathbf{E}_2^n + \mathsf{B}^n\mathbf{E}_1^n + \mathsf{C}^n\nabla\nabla u \rightharpoonup \mathsf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}) \; .$$

On the other hand, as $\mathbf{E}_2^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega; \mathrm{Sym})$ and $\mathsf{C}^n \xrightharpoonup{*} \mathsf{O}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, we have

$$\mathbf{D}_2^n \rightharpoonup \lim_n \mathsf{B}^n\mathbf{E}_1^n = \mathsf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega; \mathrm{Sym}) \; .$$

The corrector can be expressed by H-measure

Analogously, equating the terms standing by γ^2 gives:

$$\mathbf{D}_2^n = \mathsf{A}_0\mathbf{E}_2^n + \mathsf{B}^n\mathbf{E}_1^n + \mathsf{C}^n\nabla\nabla u \rightharpoonup \mathsf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym})\,.$$

On the other hand, as $\mathbf{E}_2^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega;\mathrm{Sym})$ and $\mathsf{C}^n \xrightharpoonup{*} \mathsf{O}$ in $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$, we have

$$\mathbf{D}_2^n \rightharpoonup \lim_n \mathsf{B}^n\mathbf{E}_1^n = \mathsf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym})\,.$$

Obviously, identifying the corrector of order 2 in γ requires the computation of the weak limit of $(\mathsf{B}^n\mathbf{E}_1^n)$, the product of two weakly convergent sequences.

The corrector can be expressed by H-measure

Analogously, equating the terms standing by γ^2 gives:

$$\mathbf{D}_2^n = \mathsf{A}_0\mathbf{E}_2^n + \mathsf{B}^n\mathbf{E}_1^n + \mathsf{C}^n\nabla\nabla u \rightharpoonup \mathsf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym}) \,.$$

On the other hand, as $\mathbf{E}_2^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega;\mathrm{Sym})$ and $\mathsf{C}^n \xrightharpoonup{*} \mathbf{O}$ in $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$, we have

$$\mathbf{D}_2^n \rightharpoonup \lim_n \mathsf{B}^n\mathbf{E}_1^n = \mathsf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym}) \,.$$

Obviously, identifying the corrector of order 2 in γ requires the computation of the weak limit of $(\mathsf{B}^n\mathbf{E}_1^n)$, the product of two weakly convergent sequences.

And such limits can be expressed by using H-measures.

The corrector can be expressed by H-measure

Analogously, equating the terms standing by γ^2 gives:

$$\mathbf{D}_2^n = \mathsf{A}_0\mathbf{E}_2^n + \mathsf{B}^n\mathbf{E}_1^n + \mathsf{C}^n\nabla\nabla u \rightharpoonup \mathsf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym})\,.$$

On the other hand, as $\mathbf{E}_2^n \rightharpoonup \mathbf{0}$ in $\mathrm{L}^2(\Omega;\mathrm{Sym})$ and $\mathsf{C}^n \xrightharpoonup{*} \mathsf{O}$ in $\mathrm{L}^\infty(\Omega;\mathcal{L}(\mathrm{Sym},\mathrm{Sym}))$, we have

$$\mathbf{D}_2^n \rightharpoonup \lim_n \mathsf{B}^n\mathbf{E}_1^n = \mathsf{C}_0\nabla\nabla u \quad \text{in} \quad \mathrm{L}^2(\Omega;\mathrm{Sym})\,.$$

Obviously, identifying the corrector of order 2 in γ requires the computation of the weak limit of $(\mathsf{B}^n\mathbf{E}_1^n)$, the product of two weakly convergent sequences.

And such limits can be expressed by using H-measures.

For a physical plate, we assume that Ω is a bounded region, so L^∞ weak $*$ topology is stronger than L^2 weak, and we are indeed in the situation where both sequences converge weakly in L^2 to zero.

The H-measure

Let $\tilde{\boldsymbol{\mu}}$ be the H-measure corresponding to the sequence $[\mathsf{B}^n \;\; \mathbf{E}_1^n]^\top$:

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\sigma} \\ \boldsymbol{\rho} & \boldsymbol{\nu} \end{bmatrix},$$

which is defined as a $(d^4 + d^2) \times (d^4 + d^2)$ Hermitian nonnegative matrix Radon measure.

The H-measure

Let $\tilde{\boldsymbol{\mu}}$ be the H-measure corresponding to the sequence $[\mathsf{B}^n \ \ \mathbf{E}_1^n]^\top$:

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\sigma} \\ \boldsymbol{\rho} & \boldsymbol{\nu} \end{bmatrix},$$

which is defined as a $(d^4 + d^2) \times (d^4 + d^2)$ Hermitian nonnegative matrix Radon measure.

More precisely, block $\boldsymbol{\mu}$ is the H-measure associated to (a subsequence of) (B^n), while $\boldsymbol{\sigma} = \boldsymbol{\rho}^*$ is the H-measure corresponding to the product $\mathsf{B}^n\mathbf{E}_1^n$. For simplicity, by $\mathbf{v}^n := [\mathsf{B}^n \ \ \mathbf{E}_1^n]^\top$ we denote the $(d^4 + d^2) \times 1$ column matrix, but we still use the original four indices for B^n and two for $\mathbf{E}_1^n$, avoiding explicit writing of the appropriate bijection from $\{1, \dots, d\}^4 \bigcup \{1, \dots, d\}^2$ to $\{1, \dots, d^4 + d^2\}$, as such notation will be needed again for interpretation of the limit. All indices have range in $\{1, \dots, d\}$.

The H-measure

Let $\tilde{\boldsymbol{\mu}}$ be the H-measure corresponding to the sequence $[\mathbf{B}^n \ \ \mathbf{E}_1^n]^\top$:

$$\tilde{\boldsymbol{\mu}} = \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\sigma} \\ \boldsymbol{\rho} & \boldsymbol{\nu} \end{bmatrix},$$

which is defined as a $(d^4 + d^2) \times (d^4 + d^2)$ Hermitian nonnegative matrix Radon measure.

More precisely, block $\boldsymbol{\mu}$ is the H-measure associated to (a subsequence of) $(\mathbf{B}^n)$, while $\boldsymbol{\sigma} = \boldsymbol{\rho}^*$ is the H-measure corresponding to the product $\mathbf{B}^n \mathbf{E}_1^n$. For simplicity, by $\mathbf{v}^n := [\mathbf{B}^n \ \ \mathbf{E}_1^n]^\top$ we denote the $(d^4 + d^2) \times 1$ column matrix, but we still use the original four indices for $\mathbf{B}^n$ and two for $\mathbf{E}_1^n$, avoiding explicit writing of the appropriate bijection from $\{1, \dots, d\}^4 \bigcup \{1, \dots, d\}^2$ to $\{1, \dots, d^4 + d^2\}$, as such notation will be needed again for interpretation of the limit. All indices have range in $\{1, \dots, d\}$.

After computing this limit, we write it as $\mathbf{C}_0 \nabla\nabla u$, and thus identify $\mathbf{C}_0$. Our goal is to use the localisation principle for H-measures to express that limit, i.e. the measure $\boldsymbol{\sigma}$, from the H-measure $\boldsymbol{\mu}$. To this end we need to choose certain expressions relating $\mathbf{E}_1^n$ and $\mathbf{B}^n$.

Computing the H-measure

Firstly, we insert the expansions for $\mathsf{M}^n(\cdot\,;\gamma), \mathsf{M}(\cdot\,;\gamma)$ and u^n_γ into BVP

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathsf{M}^n(\cdot\,;\gamma)\nabla\nabla u^n_\gamma) = f_\gamma = \operatorname{div}\operatorname{div}(\mathsf{M}(\cdot\,;\gamma)\nabla\nabla u) \text{ in } \Omega \\ u^n_\gamma \in \mathrm{H}^2_0(\Omega)\ , \end{cases}$$

and after comparing expressions corresponding to the first power of γ, we obtain

$$\operatorname{div}\operatorname{div}(\mathsf{A}_0\mathbf{E}^n_1 + \mathsf{B}^n\nabla\nabla u) = \operatorname{div}\operatorname{div}(\mathsf{B}_0\nabla\nabla u).$$

Computing the H-measure

Firstly, we insert the expansions for $\mathbf{M}^n(\cdot\,;\gamma), \mathbf{M}(\cdot\,;\gamma)$ and u^n_γ into BVP

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n(\cdot\,;\gamma)\nabla\nabla u^n_\gamma) = f_\gamma = \operatorname{div}\operatorname{div}(\mathbf{M}(\cdot\,;\gamma)\nabla\nabla u) \text{ in } \Omega \\ u^n_\gamma \in \mathrm{H}^2_0(\Omega)\ , \end{cases}$$

and after comparing expressions corresponding to the first power of γ, we obtain

$$\operatorname{div}\operatorname{div}(\mathbf{A}_0\mathbf{E}^n_1 + \mathbf{B}^n\nabla\nabla u) = \operatorname{div}\operatorname{div}(\mathbf{B}_0\nabla\nabla u).$$

Due to $\mathbf{B}_0 = \mathbf{O}$ we have

$$(+) \qquad \operatorname{div}\operatorname{div}(\mathbf{A}_0\mathbf{E}^n_1 + \mathbf{B}^n\nabla\nabla u) = 0\ ,$$

as well as Schwarz's symmetries:

$$(++) \qquad \partial_r\partial_s(\mathbf{E}^n_1)_{kl} - \partial_k\partial_l(\mathbf{E}^n_1)_{rs} = 0\ .$$

Additionally assume that $\nabla\nabla u$ is continuous, and apply the Localisation principle to relations $(+)$ and $(++)$.

Localisation on $(+)$

For chosen $i, j \in \{1, \ldots, d\}$, after defining matrix $\mathbf{A}^{ij} \in M_{1\times(d^4+d^2)}(\mathbf{R})$ by

$$\mathbf{A}^{ij} := \left[\mathbf{A}^{\mathsf{B}^{ij}} \ \mathbf{A}^{\mathbf{E}_1^{ij}}\right] \ ,$$

where each $\mathbf{A}^{\mathsf{B}^{ij}}$ is a $1 \times d^4$ matrix with entries

$$\left[\mathbf{A}^{\mathsf{B}^{ij}}\right]_{vwkl} := \begin{cases} \partial_k \partial_l u, & \text{if } (v, w) = (i, j) \\ 0, & \text{otherwise }, \end{cases}$$

and each $\mathbf{A}^{\mathbf{E}_1^{ij}}$ is a $1 \times d^2$ matrix with entries given by

$$\left[\mathbf{A}^{\mathbf{E}_1^{ij}}\right]_{kl} := [\mathsf{A}_0]_{ijkl}.$$

Localisation on $(+)$

For chosen $i, j \in \{1, \ldots, d\}$, after defining matrix $\mathbf{A}^{ij} \in M_{1\times(d^4+d^2)}(\mathbf{R})$ by

$$\mathbf{A}^{ij} := \left[\mathbf{A}^{\mathsf{B}^{ij}} \ \mathbf{A}^{\mathbf{E}_1^{ij}}\right] \ ,$$

where each $\mathbf{A}^{\mathsf{B}^{ij}}$ is a $1 \times d^4$ matrix with entries

$$\left[\mathbf{A}^{\mathsf{B}^{ij}}\right]_{vwkl} := \begin{cases} \partial_k \partial_l u, & \text{if } (v,w) = (i,j) \\ 0, & \text{otherwise ,} \end{cases}$$

and each $\mathbf{A}^{\mathbf{E}_1^{ij}}$ is a $1 \times d^2$ matrix with entries given by

$$\left[\mathbf{A}^{\mathbf{E}_1^{ij}}\right]_{kl} := [\mathsf{A}_0]_{ijkl}.$$

It is easy to check that the assumptions of the Localisation principle are fulfilled for $m = 2$. Therefore

$$\left(\sum_{i,j=1}^{d} (2\pi i)^2 \, \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2} \, \mathbf{A}^{ij}(\mathbf{x})\right) \tilde{\boldsymbol{\mu}} = \mathbf{0}$$

and from here we can conclude that

$$\sum_{i,j,k,l=1}^{d} \xi_i \xi_j \bar{\boldsymbol{\mu}}_{ijkl}^{pqrs} \partial_k \partial_l u + \sum_{i,j,k,l=1}^{d} \xi_i \xi_j \boldsymbol{\rho}_{pqrs}^{kl} [\mathsf{A}_0]_{ijkl} = 0 \ .$$

Localisation on $(++)$

For fixed $k, l, r, s \in \{1, \ldots, d\}$, $(k, l) \neq (r, s)$, define $\mathbf{A}^{ij} \in M_{1\times(d^4+d^2)}(\mathbf{R})$ by

$$\mathbf{A}^{ij} := \begin{bmatrix} \mathbf{0} \ \mathbf{A}^{\mathbf{E}_1^{ij}} \end{bmatrix} ,$$

where $\mathbf{A}^{\mathbf{E}_1^{ij}}$ is a $1 \times d^2$ matrix whose entries are given by

$$\left[\mathbf{A}^{\mathbf{E}_1^{ij}}\right]_{vw} = \begin{cases} 1, & \text{if } (i, j, v, w) = (r, s, k, l) \\ -1, & \text{if } (i, j, v, w) = (k, l, r, s) \\ 0, & \text{otherwise} \end{cases} .$$

Localisation on $(++)$

For fixed $k, l, r, s \in \{1, \ldots, d\}$, $(k,l) \neq (r,s)$, define $\mathbf{A}^{ij} \in M_{1\times(d^4+d^2)}(\mathbf{R})$ by

$$\mathbf{A}^{ij} := \left[\mathbf{0}\ \mathbf{A}^{\mathbf{E}_1^{ij}}\right] \ ,$$

where $\mathbf{A}^{\mathbf{E}_1^{ij}}$ is a $1 \times d^2$ matrix whose entries are given by

$$\left[\mathbf{A}^{\mathbf{E}_1^{ij}}\right]_{vw} = \begin{cases} 1, & \text{if } (i,j,v,w) = (r,s,k,l) \\ -1, & \text{if } (i,j,v,w) = (k,l,r,s) \\ 0, & \text{otherwise} \end{cases} \ .$$

Again, the Localisation principle with $m = 2$ gives us

$$\left(\sum_{i,j=1}^{d} (2\pi i)^2 \, \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2} \, \mathbf{A}^{ij}(\mathbf{x}) \right) \tilde{\boldsymbol{\mu}} = \mathbf{0} \ ,$$

which yields

$$\xi_r \xi_s \boldsymbol{\rho}^{kl}_{pqrs} = \xi_k \xi_l \boldsymbol{\rho}^{rs}_{pqrs} \ .$$

The above is trivially satisfied for $(k,l) = (r,s)$.

Combining two relations

By multiplying the relation obtained from $(+)$ by $\xi_r \xi_s$ and summing over r, s

$$\sum_{i,j,k,l,r,s=1}^{d} \xi_i \xi_j \xi_r \xi_s \bar{\boldsymbol{\mu}}_{ijkl}^{pqrs} \partial_k \partial_l u + \sum_{i,j,k,l,r,s=1}^{d} \xi_i \xi_j \xi_r \xi_s \boldsymbol{\rho}_{pqrs}^{kl} [\mathbf{A}_0]_{ijkl} = 0.$$

By using the other relation, we can rewrite it in an equivalent form

$$\sum_{i,j,k,l,r,s=1}^{d} \xi_i \xi_j \xi_r \xi_s \bar{\boldsymbol{\mu}}_{ijkl}^{pqrs} \partial_k \partial_l u + \sum_{i,j,k,l,r,s=1}^{d} \xi_i \xi_j \xi_k \xi_l \boldsymbol{\rho}_{pqrs}^{rs} [\mathbf{A}_0]_{ijkl} = 0 \ ,$$

which, after division by

$$\sum_{i,j,k,l=1}^{d} [\mathbf{A}_0]_{ijkl} \xi_i \xi_j \xi_k \xi_l = \mathbf{A}_0 (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) > 0$$

yields

$$\sum_{r,s=1}^{d} \bar{\boldsymbol{\rho}}_{pqrs}^{rs} = - \sum_{i,j,k,l,r,s=1}^{d} \frac{\xi_i \xi_j \xi_r \xi_s}{\mathbf{A}_0 (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi})} \boldsymbol{\mu}_{ijkl}^{pqrs} \partial_k \partial_l u \ .$$

Combining two relations

By multiplying the relation obtained from $(+)$ by $\xi_r \xi_s$ and summing over r, s

$$\sum_{i,j,k,l,r,s=1}^{d} \xi_i \xi_j \xi_r \xi_s \bar{\boldsymbol{\mu}}^{pqrs}_{ijkl} \partial_k \partial_l u + \sum_{i,j,k,l,r,s=1}^{d} \xi_i \xi_j \xi_r \xi_s \boldsymbol{\rho}^{kl}_{pqrs} [\mathbf{A}_0]_{ijkl} = 0.$$

By using the other relation, we can rewrite it in an equivalent form

$$\sum_{i,j,k,l,r,s=1}^{d} \xi_i \xi_j \xi_r \xi_s \bar{\boldsymbol{\mu}}^{pqrs}_{ijkl} \partial_k \partial_l u + \sum_{i,j,k,l,r,s=1}^{d} \xi_i \xi_j \xi_k \xi_l \boldsymbol{\rho}^{rs}_{pqrs} [\mathbf{A}_0]_{ijkl} = 0 \ ,$$

which, after division by

$$\sum_{i,j,k,l=1}^{d} [\mathbf{A}_0]_{ijkl} \xi_i \xi_j \xi_k \xi_l = \mathbf{A}_0(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) > 0$$

yields

$$\sum_{r,s=1}^{d} \bar{\boldsymbol{\rho}}^{rs}_{pqrs} = - \sum_{i,j,k,l,r,s=1}^{d} \frac{\xi_i \xi_j \xi_r \xi_s}{\mathbf{A}_0(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi})} \boldsymbol{\mu}^{pqrs}_{ijkl} \partial_k \partial_l u \ .$$

Recall that $\lim_n \mathbf{B}^n \mathbf{E}^n_1 = \mathbf{C}_0 \nabla\nabla u$ weakly in $L^2(\Omega)$, and thus also weak $*$ in the space of Radon measures.

Hermitian character of H-measures

As $\boldsymbol{\sigma} = \boldsymbol{\rho}^*$ is the H-measure corresponding to the product $\mathbf{B}^n \mathbf{E}_1^n$, for an arbitrary $\varphi \in \mathrm{C}_c(\Omega)$, we have in components

$$\begin{aligned}
\int\limits_{\Omega} \varphi(\mathbf{x}) \sum_{r,s=1}^{d} [\mathbf{C}_0(\mathbf{x})]_{pqrs} \partial_r \partial_s u(\mathbf{x})\, d\mathbf{x} &= \left\langle \sum_{r,s=1}^{d} [\mathbf{C}_0]_{pqrs} \partial_r \partial_s u, \varphi \right\rangle \\
&= \left\langle \sum_{r,s=1}^{d} \bar{\boldsymbol{\sigma}}_{rs}^{pqrs}, \varphi \boxtimes 1 \right\rangle \\
&= \int\limits_{\Omega \times S^{d-1}} \varphi(\mathbf{x}) d \left(\sum_{r,s=1}^{d} \bar{\boldsymbol{\sigma}}_{rs}^{pqrs} \right) (\mathbf{x}, \boldsymbol{\xi}) \\
&= \int\limits_{\Omega \times S^{d-1}} \varphi(\mathbf{x}) d \left(\sum_{r,s=1}^{d} \left(\boldsymbol{\rho}^{\top} \right)_{rs}^{pqrs} \right) (\mathbf{x}, \boldsymbol{\xi}) \\
&= \int\limits_{\Omega \times S^{d-1}} \varphi(\mathbf{x}) d \left(\sum_{r,s=1}^{d} \boldsymbol{\rho}_{pqrs}^{rs} \right) (\mathbf{x}, \boldsymbol{\xi}) \,.
\end{aligned}$$

The result

Finally, inserting the expression for $\sum_{r,s=1}^{d} \bar{\boldsymbol{\rho}}^{rs}_{pqrs}$ from before

$$\sum_{r,s=1}^{d} \int_{\Omega} [\mathbf{C}_0]_{pqrs} \varphi \partial_r \partial_s u \, d\mathbf{x} = -\int_{\Omega \times S^{d-1}} \sum_{i,j,k,l,r,s=1}^{d} \frac{\xi_i \xi_j \xi_k \xi_l}{\mathbf{A}_0(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi})} \varphi \partial_r \partial_s u \, d\boldsymbol{\mu}^{ijrs}_{pqkl}(\mathbf{x}, \boldsymbol{\xi}).$$

By varying $u \in \mathrm{C}^2(\Omega)$ (e.g. choosing $\nabla\nabla u$ constant on the support of φ), one easily deduces the result which is stated in the following theorem.

Theorem. *The tensor $\mathbf{M}(\cdot\,;\gamma)$ admits the expansion*

$$\mathbf{M}(\cdot\,;\gamma) := \mathbf{A}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2) \,,$$

where the second-order H-correction $\mathbf{C}_0 \in \mathrm{L}^{\infty}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ satisfies

$$\int_{\Omega} [\mathbf{C}_0]_{pqrs} \varphi \, d\mathbf{x} = -\sum_{i,j,k,l=1}^{d} \left\langle \boldsymbol{\mu}^{ijrs}_{pqkl}, \frac{\varphi \xi_i \xi_j \xi_k \xi_l}{\mathbf{A}_0(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi})} \right\rangle .$$

∎

Sequences not converging to zero

If we take $\mathbf{B}^n \xrightarrow{*} \mathbf{B}^0$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$ and $\mathbf{C}^n \xrightarrow{*} \mathbf{C}^0$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$, we get

$$\mathbf{M}(\cdot\,; \gamma) := \mathbf{A}_0 + \gamma \mathbf{B}^0 + \gamma^2 (\mathbf{C}^0 + \mathbf{C}_0) + o(\gamma^2),$$

where $\mathbf{C}_0$ is given in the Theorem.

Periodic case

- Let Y be the d-dimensional torus, $\mathsf{M} \in \mathrm{L}^\infty(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) \cap \mathfrak{M}_2(\alpha, \beta; Y)$
- Assume $\mathsf{M}^n(\mathbf{x}) := \mathsf{M}(n\mathbf{x}), \mathbf{x} \in \Omega \subseteq \mathbf{R}^d$ (projection of $\mathbf{R}^d$ to Y assumed)
- $\mathrm{H}^2(Y)$ consists of 1-periodic functions, with the norm taken over the fundamental period
- $\mathrm{H}^2(Y)/\mathbf{R}$ is equipped with the norm $\|\nabla\nabla \cdot\|_{\mathrm{L}^2(Y)}$
- $\mathbf{E}_{ij}, 1 \leqslant i, j \leqslant d$ are $\mathrm{M}_{d\times d}$ matrices defined as

$$[\mathbf{E}_{ij}]_{kl} = \begin{cases} 1, & \text{if } i = j = k = l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

Periodic case

- Let Y be the d-dimensional torus, $\mathbf{M} \in \mathrm{L}^\infty(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) \cap \mathfrak{M}_2(\alpha, \beta; Y)$
- Assume $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), \mathbf{x} \in \Omega \subseteq \mathbf{R}^d$ (projection of $\mathbf{R}^d$ to Y assumed)
- $\mathrm{H}^2(Y)$ consists of 1-periodic functions, with the norm taken over the fundamental period
- $\mathrm{H}^2(Y)/\mathbf{R}$ is equipped with the norm $\|\nabla\nabla \cdot \|_{\mathrm{L}^2(Y)}$
- $\mathbf{E}_{ij}, 1 \leqslant i, j \leqslant d$ are $\mathrm{M}_{d\times d}$ matrices defined as

$$[\mathbf{E}_{ij}]_{kl} = \begin{cases} 1, & \text{if } i = j = k = l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem. $(\mathbf{M}^n)$ *H-converges to a constant tensor* $\mathbf{M}^\infty \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *defined as*

$$m^\infty_{klij} = \int_Y \mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x})) : (\mathbf{E}_{kl} + \nabla\nabla w_{kl}(\mathbf{x}))\, d\mathbf{x},$$

where (w_{ij}) *is the family of unique solutions in* $\mathrm{H}^2(Y)/\mathbf{R}$ *of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x}))) = 0 \text{ in Y} \\ \mathbf{x} \to w_{ij}(\mathbf{x}) \quad \text{is } Y\text{-periodic.} \end{cases}$$

∎

Small-amplitude assumptions

Theorem. *Let* $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym};\mathrm{Sym})$ *be a constant coercive tensor,* $\mathbf{B}^n(\mathbf{x}) := \mathbf{B}(n\mathbf{x})$, $\mathbf{x} \in \Omega$, *where* $\Omega \subseteq \mathbf{R}^d$ *is a bounded, open set, and* $\mathbf{B}$ *is a Y-periodic,* L^∞ *tensor function, satisfying* $\int_Y \mathbf{B}(\mathbf{x})\,d\mathbf{x} = 0$. *Then*

$$\mathbf{M}^n_\gamma(\mathbf{x}) := \mathbf{A}_0 + \gamma\mathbf{B}^n(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

H-converges (for any small γ*) to a tensor* $\mathbf{M}_\gamma := \mathbf{A}_0 + \gamma^2\mathbf{C}_0 + o(\gamma^2)$,

Small-amplitude assumptions

Theorem. *Let* $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym};\mathrm{Sym})$ *be a constant coercive tensor,* $\mathbf{B}^n(\mathbf{x}) := \mathbf{B}(n\mathbf{x})$, $\mathbf{x} \in \Omega$, *where* $\Omega \subseteq \mathbf{R}^d$ *is a bounded, open set, and* $\mathbf{B}$ *is a Y-periodic,* L^∞ *tensor function, satisfying* $\int_Y \mathbf{B}(\mathbf{x})\, d\mathbf{x} = 0$*. Then*

$$\mathbf{M}^n_\gamma(\mathbf{x}) := \mathbf{A}_0 + \gamma \mathbf{B}^n(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

H-converges (for any small γ*) to a tensor* $\mathbf{M}_\gamma := \mathbf{A}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2)$ *, where*

$$\begin{aligned}\mathbf{C}_0 \mathbf{E}_{mn} : \mathbf{E}_{rs} = (2\pi i)^2 \sum_{\mathbf{k}\in J} a^{mn}_{-\mathbf{k}} \mathbf{B}_{\mathbf{k}} (\mathbf{k}\otimes\mathbf{k}) : \mathbf{E}_{rs} + \\ + (2\pi i)^4 \sum_{\mathbf{k}\in J} a^{mn}_{\mathbf{k}} a^{rs}_{-\mathbf{k}} \mathbf{A}_0 (\mathbf{k}\otimes\mathbf{k}) : \mathbf{k}\otimes\mathbf{k} + \\ + (2\pi i)^2 \sum_{\mathbf{k}\in J} a^{rs}_{-\mathbf{k}} \mathbf{B}_{\mathbf{k}} \mathbf{E}_{mn} : \mathbf{k}\otimes\mathbf{k}\,,\end{aligned}$$

with $m, n, r, s \in \{1, 2, \cdots, d\}$, $J := \mathbf{Z}^d \setminus \{0\}$*, and*

$$a^{mn}_{\mathbf{k}} = -\frac{\mathbf{B}_{\mathbf{k}} \mathbf{E}_{mn} \mathbf{k}\cdot\mathbf{k}}{(2\pi i)^2 \mathbf{A}_0 (\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})}, \quad \mathbf{k} \in J,$$

and $\mathbf{B}_{\mathbf{k}}$ *are the Fourier coefficients of function* $\mathbf{B}$*.* ∎

Result by applying H-measures

The corresponding H-measure of the sequence (B^n) can be explicitly computed

$$\boldsymbol{\mu}_{ijrs}^{pqkl} = \lambda(\mathbf{x}) \sum_{\mathbf{k}\in\mathbf{Z}^d} [\bar{\mathsf{B}}_{\mathbf{k}}]_{pqkl} [\mathsf{B}_{\mathbf{k}}]_{ijrs}\, \delta_{\frac{\mathbf{k}}{|\mathbf{k}|}}(\boldsymbol{\xi}),$$

where λ denotes the Lebesgue measure on $\mathbf{R}^d$ and $\mathsf{B}_{\mathbf{k}}$, $\mathbf{k} \in \mathbf{Z}^d$, are Fourier coefficients of function B. After inserting this expression in the formula in the Theorem, we can easily calculate C_0 explicitly:

$$\mathsf{C}_0 = -\sum_{\mathbf{k}\in\mathbf{Z}^d} \frac{\mathsf{B}_{\mathbf{k}}(\mathbf{k}\otimes\mathbf{k}) \otimes \mathsf{B}_{\mathbf{k}}^\top(\mathbf{k}\otimes\mathbf{k})}{\mathsf{A}_0(\mathbf{k}\otimes\mathbf{k}) : (\mathbf{k}\otimes\mathbf{k})},$$

where the tensor product of two matrices $\mathbf{A}, \mathbf{B} \in M_d(\mathbf{C})$ is the fourth-order tensor with entries

$$[\mathbf{A}\otimes\mathbf{B}]_{ijkl} = a_{ij}\bar{b}_{kl}.$$

This coincides with the result obtained via explicit formula for the homogenisation limit of a periodic sequence of tensors describing material properties in the Kirchhoff-Love model.

Vibrating plate equation

Take $\Omega \subseteq \mathbf{R}^d$ a bounded open set, $T > 0$ and $\rho_- > 0$, and denote $V := \mathrm{H}_0^2(\Omega)$, $H := \mathrm{L}^2(\Omega)$, while $V' = \mathrm{H}^{-2}(\Omega)$.
For given $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ and $\rho \in \mathrm{L}^\infty(\Omega; [\rho_-, \rho_+])$, as well as $v \in V$, $w \in H$ and $f \in \mathrm{L}^1(0, T; H)$ on the right-hand side, consider the initial–boundary value problem:

$$\begin{cases} \rho u'' + \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \\ u(0, \cdot) = v \\ u'(0, \cdot) = w \end{cases},$$

where we seek $u \in \mathrm{L}^2(0, T; V)$ such that $u' \in \mathrm{L}^2(0, T; H)$.

Vibrating plate equation

Take $\Omega \subseteq \mathbf{R}^d$ a bounded open set, $T > 0$ and $\rho_- > 0$, and denote $V := \mathrm{H}^2_0(\Omega)$, $H := \mathrm{L}^2(\Omega)$, while $V' = \mathrm{H}^{-2}(\Omega)$.
For given $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ and $\rho \in \mathrm{L}^\infty(\Omega; [\rho_-, \rho_+])$, as well as $v \in V$, $w \in H$ and $f \in \mathrm{L}^1(0, T; H)$ on the right-hand side, consider the initial–boundary value problem:

$$\left\{ \begin{aligned} \rho u'' + \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) &= f \\ u(0, \cdot) &= v \\ u'(0, \cdot) &= w \end{aligned} \right. ,$$

where we seek $u \in \mathrm{L}^2(0, T; V)$ such that $u' \in \mathrm{L}^2(0, T; H)$.
This problem has a unique solution satisfying $u'' \in \mathrm{L}^1(0, T; V')$ as well. In fact, the solution belongs to the space $\mathrm{C}([0, T]; V)$, with $u' \in \mathrm{C}([0, T]; H)$, and satisfies the estimate

$$(\forall\, t \in [0, T]) \quad \|u(t)\|_V + \|u'(t)\|_H \leqslant C \ ,$$

where the constant C depends on $\alpha, \beta, \rho_-, \rho_+, f, v, w$.

Homogenisation

Consider a sequence of such problems and let us show that the limit of their solutions satisfies an analogous equation. In fact, one has

Theorem. *Assume that* (ρ^n) *and* $(\mathbf{M}^n)$ *are sequences in* $\mathrm{L}^\infty(\Omega;[\rho_-,\rho_+])$ *and* $\mathfrak{M}_2(\alpha,\beta;\Omega)$ *respectively, such that*

$$\rho^n \xrightarrow{\mathrm{L}^\infty(\Omega)*} \rho^\infty \qquad \textit{and} \qquad \mathbf{M}^n \xrightarrow{H} \mathbf{M}^\infty \, .$$

Homogenisation

Consider a sequence of such problems and let us show that the limit of their solutions satisfies an analogous equation. In fact, one has

Theorem. *Assume that* (ρ^n) *and* $(\mathbf{M}^n)$ *are sequences in* $\mathrm{L}^\infty(\Omega;[\rho_-,\rho_+])$ *and* $\mathfrak{M}_2(\alpha,\beta;\Omega)$ *respectively, such that*

$$\rho^n \xrightharpoonup{\mathrm{L}^\infty(\Omega)*} \rho^\infty \qquad \textit{and} \qquad \mathbf{M}^n \xrightharpoonup{H} \mathbf{M}^\infty \ .$$

Let u_n *be the solution of the initial boundary value problem*

$$\left\{ \begin{aligned} \rho^n u_n'' + \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) &= f_n \\ u_n(0,\cdot) &= v_n \\ \rho^n u_n'(0,\cdot) &= w_n \end{aligned} \right. \quad ,$$

with boundary conditions given by $u_n \in \mathrm{L}^2([0,T];V)$ *and* $u_n' \in \mathrm{L}^2([0,T];H)$*, where we assume that* $v_n \rightharpoonup v_\infty$ *in* V*, and* $w_n \rightharpoonup w_\infty$ *in* H*; the forcing term* f_n *we take from a bounded set in the space* $\mathrm{L}^2(0,T;H)$*, assuming* $f_n \rightharpoonup f_\infty$*.*

Homogenisation

Consider a sequence of such problems and let us show that the limit of their solutions satisfies an analogous equation. In fact, one has

Theorem. *Assume that* (ρ^n) *and* $(\mathbf{M}^n)$ *are sequences in* $\mathrm{L}^\infty(\Omega;[\rho_-,\rho_+])$ *and* $\mathfrak{M}_2(\alpha,\beta;\Omega)$ *respectively, such that*

$$\rho^n \xrightarrow{\mathrm{L}^\infty(\Omega)*} \rho^\infty \qquad \textit{and} \qquad \mathbf{M}^n \xrightarrow{H} \mathbf{M}^\infty .$$

Let u_n *be the solution of the initial boundary value problem*

$$\left\{ \begin{aligned} \rho^n u_n'' + \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) &= f_n \\ u_n(0,\cdot) &= v_n \\ \rho^n u_n'(0,\cdot) &= w_n \end{aligned} \right. \quad ,$$

with boundary conditions given by $u_n \in \mathrm{L}^2([0,T];V)$ *and* $u_n' \in \mathrm{L}^2([0,T];H)$*, where we assume that* $v_n \rightharpoonup v_\infty$ *in* V*, and* $w_n \rightharpoonup w_\infty$ *in* H*; the forcing term* f_n *we take from a bounded set in the space* $\mathrm{L}^2(0,T;H)$*, assuming* $f_n \rightharpoonup f_\infty$*.*

Then we have

$$u_n \xrightarrow{*} u_\infty \quad \text{in } \mathrm{L}^\infty(0,T;V) \qquad \text{and} \qquad u_n' \xrightarrow{*} u_\infty' \quad \text{in } \mathrm{L}^\infty(0,T;H),$$

where u_∞ *is the solution of the above problem for* $n=\infty$*.* ∎

Sketch of the proof

Take $\varphi \in \mathrm{C}_c^\infty(\langle 0, T\rangle)$ and define $U_n(\mathbf{x}) := \int_0^T u_n(t, \mathbf{x})\varphi(t)\, dt$ (and the same for $n = \infty$). Clearly (by choosing test functions of the form $\varphi \boxtimes \psi$)

$$U_n \rightharpoonup U_\infty \quad \text{in } \mathrm{H}^2_{\mathrm{loc}}(\Omega) \; .$$

Which equation does U_∞ satisfy?

Sketch of the proof

Take $\varphi \in \mathrm{C}_c^\infty(\langle 0, T\rangle)$ and define $U_n(\mathbf{x}) := \int_0^T u_n(t,\mathbf{x})\varphi(t)\,dt$ (and the same for $n = \infty$). Clearly (by choosing test functions of the form $\varphi \boxtimes \psi$)

$$U_n \longrightarrow U_\infty \quad \text{in } \mathrm{H}^2_{\mathrm{loc}}(\Omega)\ .$$

Which equation does U_∞ satisfy?

By multiplying the equation with φ and integrating

$$\rho^n \int_0^T u_n''\varphi\,dt + \mathsf{div}\,\mathsf{div}\left(\mathbf{M}^n \nabla\nabla \int_0^T u_n\varphi\,dt\right) = \int_0^T f_n\varphi\,dt\ ,$$

therefore $\mathsf{div}\,\mathsf{div}\left(\mathbf{M}^n\nabla\nabla U_n\right) = g_n$, where we take

$$g_n(\mathbf{x}) := \int_0^T \varphi(t) f_n(t,\mathbf{x})\,dt - \rho^n(\mathbf{x})\int_0^T u_n(t,\mathbf{x})\varphi''(t)\,dt\ .$$

Defining g_∞ as g_n, with $n = \infty$, we have the convergence $g_n \longrightarrow g_\infty$ in $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$. Indeed, for the first integral just take test functions of the form $\varphi \boxtimes \psi$, while the second is a consequence of compact embedding.

Sketch of the proof (cont.)

Using the H-convergence of $\mathbf{M}^n$

$$\mathbf{M}^n \nabla\nabla U_n \rightharpoonup \mathbf{M}^\infty \nabla\nabla U_\infty \qquad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; M_{d\times d}) \ ,$$

so by varying φ one has

$$\mathbf{M}^n \nabla\nabla u_n \rightharpoonup \mathbf{M}^\infty \nabla\nabla u_\infty \qquad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\langle 0, T\rangle \times \Omega; M_{d\times d}) \ ,$$

We can now pass to the limit in the variational form of the equation, using the earlier mentioned form of compactness by compensation, thus obtaining the claim.

A class of symbols (L. Tartar)

Actually, we can consider more general operators than $\mathcal{A}_a$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|b_m\|_{\mathrm{C}_0(\mathbf{R}^d)} = k < \infty$.

A class of symbols (L. Tartar)

Actually, we can consider more general operators than $\mathcal{A}_a$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|b_m\|_{\mathrm{C}_0(\mathbf{R}^d)} = k < \infty$.

To such a symbol s, a *standard operator* $S_s \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$ is assigned by

$$S_s = \sum_m \mathcal{A}_{\alpha_m} M_b \ ,$$

with $\|S_s\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$.

A class of symbols (L. Tartar)

Actually, we can consider more general operators than $\mathcal{A}_a$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|b_m\|_{\mathrm{C}_0(\mathbf{R}^d)} = k < \infty$.

To such a symbol s, a *standard operator* $S_s \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$ is assigned by

$$S_s = \sum_m \mathcal{A}_{\alpha_m} M_b \ ,$$

with $\|S_s\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$.

Clearly, S_s does not depend on the above decomposition, as

$$\widehat{S_s u}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} s(\mathbf{x}, \boldsymbol{\xi}/|\boldsymbol{\xi}|) u(\mathbf{x}) \, d\mathbf{x} \ ,$$

for u in a dense set of $\mathrm{L}^2(\mathbf{R}^d)$ (e.g. $\mathcal{S}$).

A class of symbols (cont.)

Any operator $A \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, which differs from S_s only by a compact operator, is an *operator of symbol* s, like

$$L_s = \sum_m M_{b_m} \mathcal{A}_{\alpha_m} \ ,$$

where $\|L_s\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$. Neither L_s depends on the decomposition.

A class of symbols (cont.)

Any operator $A \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, which differs from S_s only by a compact operator, is an *operator of symbol* s, like

$$L_s = \sum_m M_{b_m} \mathcal{A}_{\alpha_m} \ ,$$

where $\|L_s\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$. Neither L_s depends on the decomposition.

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^r)$*, then there exists its subsequence and an H-measure* $\boldsymbol{\mu}$*, which is a Hermitian non-negative* $r \times r$ *matrix of distributions of order zero on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and any operators* $L_{s_1}, L_{s_2} \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$*, with symbols* s_1, s_2 *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} L_{s_1}(\varphi_1 u^j_{n'}) \overline{L_{s_2}(\varphi_2 u^k_{n'})} \, d\boldsymbol{\xi} = \langle \mu^{jk}, \varphi_1 s_1 \overline{\varphi_2 s_2} \rangle \ .$$

∎

A class of symbols (cont.)

Any operator $A \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, which differs from S_s only by a compact operator, is an *operator of symbol* s, like

$$L_s = \sum_m M_{b_m} \mathcal{A}_{\alpha_m} \ ,$$

where $\|L_s\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$. Neither L_s depends on the decomposition.

Theorem. *If* $\mathrm{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^r)$, *then there exists its subsequence and an H-measure* $\boldsymbol{\mu}$, *which is a Hermitian non-negative* $r \times r$ *matrix of distributions of order zero on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and any operators* $L_{s_1}, L_{s_2} \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, *with symbols* s_1, s_2 *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} L_{s_1}(\varphi_1 u^j_{n'}) \overline{L_{s_2}(\varphi_2 u^k_{n'})} \, d\boldsymbol{\xi} = \langle \mu^{jk}, \varphi_1 s_1 \overline{\varphi_2 s_2} \rangle \ .$$

∎

P. Gérard used a different approach, by using classical symbols. However, it is important to have symbols of lower regularity, as they come in applications from coefficients in PDEs.

A class of symbols (cont.)

Any operator $A \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, which differs from S_s only by a compact operator, is an *operator of symbol* s, like

$$L_s = \sum_m M_{b_m} \mathcal{A}_{\alpha_m} \ ,$$

where $\|L_s\|_{\mathcal{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$. Neither L_s depends on the decomposition.

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^r)$, *then there exists its subsequence and an H-measure* $\boldsymbol{\mu}$, *which is a Hermitian non-negative* $r \times r$ *matrix of distributions of order zero on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and any operators* $L_{s_1}, L_{s_2} \in \mathcal{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, *with symbols* s_1, s_2 *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} L_{s_1}(\varphi_1 u^j_{n'}) \overline{L_{s_2}(\varphi_2 u^k_{n'})} \, d\boldsymbol{\xi} = \langle \mu^{jk}, \varphi_1 s_1 \overline{\varphi_2 s_2} \rangle \ .$$

∎

P. Gérard used a different approach, by using classical symbols. However, it is important to have symbols of lower regularity, as they come in applications from coefficients in PDEs.

We can consider $\Omega \subseteq \mathbf{R}^d$ as a domain, or even a manifold (with a volume form).

Symmetric systems

$$\sum_k \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \ \ , \ \mathbf{A}^k \text{ Hermitian}$$

Symmetric systems

$$\sum_k \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \ \ , \ \mathbf{A}^k \text{ Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightharpoonup{\mathrm{L}^2} 0 \quad \text{(weakly)},$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

Symmetric systems

$$\sum_k \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \ , \ \mathbf{A}^k \ \text{Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0 \quad \text{(weakly)},$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

If supports of $\mathsf{u}^n, \mathsf{f}^n$ are contained inside Ω, we can extend them by zero to $\mathbf{R}^d$.

Theorem. (localisation property) *If* $\mathsf{u}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)^r$ *defines* $\boldsymbol{\mu}$, *and if* u^n *satisfies:*

$$\partial_k\left(\mathbf{A}^k \mathsf{u}^n\right) \to 0 \ \text{ in the space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\Omega \times S^{d-1}$ *it holds:*

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}^\top = \mathbf{0} \ .$$

∎

Symmetric systems

$$\sum_k \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad , \ \mathbf{A}^k \text{ Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0 \qquad \text{(weakly)},$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \qquad \text{(strongly)}.$$

If supports of $\mathsf{u}^n, \mathsf{f}^n$ are contained inside Ω, we can extend them by zero to $\mathbf{R}^d$.

Theorem. **(localisation property)** *If* $\mathsf{u}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)^r$ *defines* $\boldsymbol{\mu}$*, and if* u^n *satisfies:*

$$\partial_k\left(\mathbf{A}^k \mathsf{u}^n\right) \to 0 \ \text{ in the space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\Omega \times S^{d-1}$ *it holds:*

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi})\boldsymbol{\mu}^\top = \mathbf{0} \ .$$

∎

Thus, the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{P}$ is a singular matrix.)

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \le m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi}\ .$$

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \leq m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi}\ .$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish in infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \leq m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi}\ .$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish in infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).
On the other hand, $\mathrm{H}^s(\mathbf{R}^d) \subseteq X^m$, for $s > m + \frac{d}{2}$.

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \le m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi}\ .$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish in infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).
On the other hand, $\mathrm{H}^s(\mathbf{R}^d) \subseteq X^m$, for $s > m + \frac{d}{2}$.
X^m is an algebra with respect to the multiplication of functions; it holds:

$$\|f * g\|_{\mathrm{L}^1} \le \|f\|_{\mathrm{L}^1} \|g\|_{\mathrm{L}^1}$$

$$\|\hat{f} \cdot \hat{g}\|_{X^0} \le \|\hat{f}\|_{X^0} \|\hat{g}\|_{X^0}$$

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \le m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi}\ .$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish in infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).
On the other hand, $\mathrm{H}^s(\mathbf{R}^d) \subseteq X^m$, for $s > m + \frac{d}{2}$.
X^m is an algebra with respect to the multiplication of functions; it holds:

$$\|f * g\|_{\mathrm{L}^1} \le \|f\|_{\mathrm{L}^1} \|g\|_{\mathrm{L}^1}$$
$$\|\hat{f} \cdot \hat{g}\|_{X^0} \le \|\hat{f}\|_{X^0} \|\hat{g}\|_{X^0}$$

$X^m_{\mathrm{loc}}(\Omega)$: the space of all functions u such that $\varphi u \in X^m$, for $\varphi \in \mathrm{C}_c^\infty(\Omega)$.

Second commutation lemma

$$X^m := \left\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^d)\ |\boldsymbol{\alpha}| \le m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \right\}$$

is a Banach space with the norm:

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \left(1 + 4\pi^2 |\boldsymbol{\xi}|^2\right)^{m/2} \left|\hat{w}(\boldsymbol{\xi})\right| d\boldsymbol{\xi}\ .$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish in infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).
On the other hand, $\mathrm{H}^s(\mathbf{R}^d) \subseteq X^m$, for $s > m + \frac{d}{2}$.
X^m is an algebra with respect to the multiplication of functions; it holds:

$$\|f * g\|_{\mathrm{L}^1} \le \|f\|_{\mathrm{L}^1} \|g\|_{\mathrm{L}^1}$$
$$\|\hat{f} \cdot \hat{g}\|_{X^0} \le \|\hat{f}\|_{X^0} \|\hat{g}\|_{X^0}$$

$X^m_{\mathrm{loc}}(\Omega)$: the space of all functions u such that $\varphi u \in X^m$, for $\varphi \in \mathrm{C}_c^\infty(\Omega)$.

Lemma. *Let $\mathcal{A}_\alpha$, M_b be standard operators, with symbols α, b, such that $\alpha \in \mathrm{C}^1(S^{d-1})$ and $b \in X^1$.*
Then $C := [\mathcal{A}_\alpha, M_b] \in \mathcal{L}\big(\mathrm{L}^2(\mathbf{R}^d), \mathrm{H}^1(\mathbf{R}^d)\big)$, and ∇C has a symbol $(\nabla_{\boldsymbol{\xi}} \alpha \cdot \nabla_{\mathbf{x}} b)\boldsymbol{\xi}$.
(we extend α to a homogeneous function on $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{\mathbf{0}\}$)

A smaller class of symbols (L. Tartar)

Corollary. *Under the above assumptions,*

$$\mathcal{A}_\alpha M_b \partial_j u = M_b \partial_j (\mathcal{A}_\alpha u) + Lu, \quad u \in \mathrm{L}^2(\mathbf{R}^d) \ ,$$

where L has a symbol $\xi_j\{\alpha, b\}$. ∎

A smaller class of symbols (L. Tartar)

Corollary. *Under the above assumptions,*

$$\mathcal{A}_\alpha M_b \partial_j u = M_b \partial_j (\mathcal{A}_\alpha u) + Lu, \quad u \in \mathrm{L}^2(\mathbf{R}^d) \ ,$$

where L has a symbol $\xi_j\{\alpha, b\}$. ∎

Actually, we can consider more general operators than $\mathcal{A}_\alpha$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}^1(\mathrm{S}^{d-1})} \|b_m\|_{X^1} < \infty$,

A smaller class of symbols (L. Tartar)

Corollary. *Under the above assumptions,*

$$\mathcal{A}_\alpha M_b \partial_j u = M_b \partial_j (\mathcal{A}_\alpha u) + Lu, \quad u \in \mathrm{L}^2(\mathbf{R}^d) \;,$$

where L has a symbol $\xi_j\{\alpha, b\}$. ∎

Actually, we can consider more general operators than $\mathcal{A}_\alpha$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \;,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}^1(\mathrm{S}^{d-1})} \|b_m\|_{X^1} < \infty$,
and *standard operators* $S_s = \sum_m \mathcal{A}_{\alpha_m} M_b$.

A smaller class of symbols (L. Tartar)

Corollary. *Under the above assumptions,*

$$\mathcal{A}_\alpha M_b \partial_j u = M_b \partial_j (\mathcal{A}_\alpha u) + Lu, \quad u \in \mathrm{L}^2(\mathbf{R}^d) \ ,$$

where L has a symbol $\xi_j \{\alpha, b\}$. ∎

Actually, we can consider more general operators than $\mathcal{A}_\alpha$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}^1(\mathrm{S}^{d-1})} \|b_m\|_{X^1} < \infty$,
and *standard operators* $S_s = \sum_m \mathcal{A}_{\alpha_m} M_b$.

Lemma. *If S_1, S_2 are standard operators with symbols s_1, s_2 as above, then*

$$\frac{\partial}{\partial x^j} [S_1, S_2] \quad \textit{has symbol} \quad \xi_j \{s_1, s_2\} \ .$$

∎

Propagation property for symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad , \qquad \mathbf{A}^k \text{ Hermitian}$$

Propagation property for symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad , \qquad \mathbf{A}^k \text{ Hermitian}$$

Theorem. *Let* $\mathbf{A}^k \in \mathrm{C}^1_0(\Omega; \mathrm{M}_{r\times r})$*.*
If $(\mathsf{u}^n, \mathsf{f}^n)$ *satisfy the above for* $n \in \mathbf{N}$*, and* $\mathsf{u}^n, \mathsf{f}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\Omega)$*, then for any* $\psi \in \mathrm{C}^1_0(\Omega \times S^{d-1})$*, the H-measure associated to sequence* $(\mathsf{u}^n, \mathsf{f}^n)$*:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix},$$

Propagation property for symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad , \qquad \mathbf{A}^k \text{ Hermitian}$$

Theorem. *Let* $\mathbf{A}^k \in \mathrm{C}_0^1(\Omega; \mathrm{M}_{r\times r})$.
If $(\mathsf{u}^n, \mathsf{f}^n)$ *satisfy the above for* $n \in \mathbf{N}$, *and* $\mathsf{u}^n, \mathsf{f}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\Omega)$,
then for any $\psi \in \mathrm{C}_0^1(\Omega \times S^{d-1})$, *the H-measure associated to sequence* $(\mathsf{u}^n, \mathsf{f}^n)$*:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix},$$

satisfies:

$$\left\langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k - 2\psi \mathbf{S} \right\rangle + \langle 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}, \psi \rangle = 0 \ ,$$

where $\mathbf{S} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^*)$, *while the Poisson bracket is:*
$\{\phi, Q\} = \nabla_{\boldsymbol{\xi}} \phi \cdot \nabla_{\mathbf{x}} Q - \nabla_{\mathbf{x}} \phi \cdot \nabla_{\boldsymbol{\xi}} Q$. *[Recall:* $\mathbf{P} = \xi_k \mathbf{A}^k$ *]* ∎

Propagation property for symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \;\; , \qquad \mathbf{A}^k \text{ Hermitian}$$

Theorem. *Let* $\mathbf{A}^k \in \mathrm{C}_0^1(\Omega; \mathrm{M}_{r\times r})$*.*
If $(\mathsf{u}^n, \mathsf{f}^n)$ *satisfy the above for* $n \in \mathbf{N}$*, and* $\mathsf{u}^n, \mathsf{f}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\Omega)$*,*
then for any $\psi \in \mathrm{C}_0^1(\Omega \times S^{d-1})$*, the H-measure associated to sequence* $(\mathsf{u}^n, \mathsf{f}^n)$*:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix},$$

satisfies:

$$\left\langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k - 2\psi \mathbf{S} \right\rangle + \langle 2 \mathsf{Re}\, \mathsf{tr} \boldsymbol{\mu}_{12}, \psi \rangle = 0 \;,$$

where $\mathbf{S} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^*)$*, while the Poisson bracket is:*
$\{\phi, Q\} = \nabla_{\boldsymbol{\xi}} \phi \cdot \nabla_{\mathbf{x}} Q - \nabla_{\mathbf{x}} \phi \cdot \nabla_{\boldsymbol{\xi}} Q.$ *[Recall:* $\mathbf{P} = \xi_k \mathbf{A}^k$ *]* ∎

$\boldsymbol{\mu}$ is associated to the pair of sequences $(\mathsf{u}^n, \mathsf{f}^n)$, the block $\boldsymbol{\mu}_{11}$ is determined by u^n, $\boldsymbol{\mu}_{22}$ with f^n, while the non-diagonal blocks correspond to the product of u^n and f^n.

The equation for H-measure

Corollary. *In the sense of distributions on $\Omega \times S^{d-1}$ the H-measure $\boldsymbol{\mu}$ satisfies:*

$$\begin{aligned}\partial^l \mathbf{P} \cdot \partial_l \boldsymbol{\mu}_{11} - \partial_t^l(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11})\xi^l \\ + (2\mathbf{S} - \partial_l \mathbf{A}^l) \cdot \boldsymbol{\mu}_{11} = 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12} \ ,\end{aligned}$$

where $\partial_t^l := \partial^l - \xi^l \xi_k \partial^k$ is the tangential gradient on the unit sphere. ∎

The equation for H-measure

Corollary. *In the sense of distributions on $\Omega \times S^{d-1}$ the H-measure $\boldsymbol{\mu}$ satisfies:*

$$\partial^l \mathbf{P} \cdot \partial_l \boldsymbol{\mu}_{11} - \partial^l_t (\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) \xi^l$$
$$+ (2\mathbf{S} - \partial_l \mathbf{A}^l) \cdot \boldsymbol{\mu}_{11} = 2 \mathrm{Re}\, \mathrm{tr} \boldsymbol{\mu}_{12} \,,$$

where $\partial^l_t := \partial^l - \xi^l \xi_k \partial^k$ is the tangential gradient on the unit sphere. ∎

This allows us to investigate the behaviour of H-measures as solutions of initial-value problems, with appropriate initial conditions.

The equation for H-measure

Corollary. *In the sense of distributions on $\Omega \times S^{d-1}$ the H-measure $\boldsymbol{\mu}$ satisfies:*

$$\partial^l \mathbf{P} \cdot \partial_l \boldsymbol{\mu}_{11} - \partial_t^l (\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) \xi^l$$
$$+ (2\mathbf{S} - \partial_l \mathbf{A}^l) \cdot \boldsymbol{\mu}_{11} = 2 \mathrm{Re}\, \mathrm{tr} \boldsymbol{\mu}_{12} \,,$$

where $\partial_t^l := \partial^l - \xi^l \xi_k \partial^k$ is the tangential gradient on the unit sphere. ∎

This allows us to investigate the behaviour of H-measures as solutions of initial-value problems, with appropriate initial conditions.
Besides the wave equations, there are applications to Maxwell's and Dirac's systems, even to the equations that change their type (like the Tricomi equation).

The wave equation

$$(\rho u')' - \mathsf{div}\,(\mathbf{A}\nabla u) = g \; .$$

The wave equation

$$(\rho u')' - \mathsf{div}\,(\mathbf{A}\nabla u) = g\ .$$

It can be written as an equivalent symmetric system ($t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$):

$$\partial_0(\rho\partial_0 u) - \sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) = g\ .$$

The wave equation

$$(\rho u')' - \mathsf{div}\,(\mathbf{A}\nabla u) = g\ .$$

It can be written as an equivalent symmetric system ($t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$):

$$\partial_0(\rho\partial_0 u) - \sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) = g\ .$$

By introducing: $v_j := \partial_j u$, for $j \in 0..d$, we obtain (Schwarz' symmetries!):

$$\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A} & \\ 0 & & & \end{bmatrix} \partial_0 \mathsf{v} + \sum_{i=1}^{d} \begin{bmatrix} 0 & -a^{i1} & \cdots & -a^{id} \\ -a^{i1} & & & \\ \vdots & & \mathbf{0} & \\ -a^{id} & & & \end{bmatrix} \partial_i \mathsf{v}$$

$$+ \begin{bmatrix} b^0 & b^1 & \cdots & b^d \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{v} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix}\ .$$

The wave equation

$$(\rho u')' - \mathsf{div}\,(\mathbf{A}\nabla u) = g\ .$$

It can be written as an equivalent symmetric system ($t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$):

$$\partial_0(\rho\partial_0 u) - \sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) = g\ .$$

By introducing: $v_j := \partial_j u$, for $j \in 0..d$, we obtain (Schwarz' symmetries!):

$$\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A} & \\ 0 & & & \end{bmatrix} \partial_0 \mathsf{v} + \sum_{i=1}^{d} \begin{bmatrix} 0 & -a^{i1} & \cdots & -a^{id} \\ -a^{i1} & & & \\ \vdots & & \mathbf{0} & \\ -a^{id} & & & \end{bmatrix} \partial_i \mathsf{v}$$

$$+ \begin{bmatrix} b^0 & b^1 & \cdots & b^d \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{v} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix}\ .$$

The symbol of differential operator is:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \xi_k \mathbf{A}^k(\mathbf{x}) = \begin{bmatrix} \xi_0 \rho & -(\mathbf{A}\boldsymbol{\xi}')^\top \\ -\mathbf{A}\boldsymbol{\xi}' & \xi_0 \mathbf{A} \end{bmatrix}\ .$$

Transport of H-measures associated to the wave equation

From the localisation property we can conclude that $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$.

Transport of H-measures associated to the wave equation

From the localisation property we can conclude that $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$. For the right hand side of the equation we have:

$$\langle \gamma, \varphi_1 \bar{\varphi}_2 \psi \rangle := \lim_n \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 v_{0,n}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 g_n}(\boldsymbol{\xi})} \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} \; .$$

From the localisation property we can conclude that $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$. For the right hand side of the equation we have:

$$\langle \gamma, \varphi_1 \bar{\varphi}_2 \psi \rangle := \lim_n \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 v_{0,n}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 g_n}(\boldsymbol{\xi})} \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} \ .$$

Theorem. *On $\mathbf{R}^{d+1} \times S^d$ measure ν satisfies ($Q := \rho\xi_0^2 - \mathbf{A}\boldsymbol{\xi}' \cdot \boldsymbol{\xi}'$):*

$$\nabla_{\boldsymbol{\xi}} Q \cdot \nabla_{\mathbf{x}}(\xi_0 \nu) - Q \partial_0 \nu + \big(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \mathbf{I}\big) \nabla_{\mathbf{x}} Q \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)\big(\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}\big)(\xi_0 \nu) = 2\mathrm{Re}\, \gamma$$

∎

From the localisation property we can conclude that $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$. For the right hand side of the equation we have:

$$\langle \gamma, \varphi_1 \bar{\varphi}_2 \psi \rangle := \lim_n \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 v_{0,n}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 g_n}(\boldsymbol{\xi})} \psi \left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right) d\boldsymbol{\xi} \, .$$

Theorem. *On $\mathbf{R}^{d+1} \times S^d$ measure ν satisfies ($Q := \rho\xi_0^2 - \mathbf{A}\boldsymbol{\xi}' \cdot \boldsymbol{\xi}'$):*

$$\nabla_{\boldsymbol{\xi}} Q \cdot \nabla_{\mathbf{x}}(\xi_0 \nu) - Q \partial_0 \nu + \big(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \mathbf{I}\big) \nabla_{\mathbf{x}} Q \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)\big(\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}\big)(\xi_0 \nu) = 2\mathrm{Re}\, \gamma$$

∎

The equation can be written in a nicer form:

$$\{Q, \xi_0 \nu\} + (\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}) \big[\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}}(\xi_0 \nu) + (d+2)(\xi_0 \nu)\big] - Q \partial_0 \nu = 2\mathrm{Re}\, \gamma \, .$$

Recent Tartar's result (2017)

Theorem. *Let* $u^n \in \mathrm{C}^0(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{C}^1(0,T;\mathrm{L}^2(\Omega))$ *be a sequence of solutions of a "wave equation"*

$$(\rho(u^n)')' - \mathsf{div}\,(\mathbf{A}\nabla u^n) + S^k \partial_k u^n \longrightarrow 0 \qquad \text{in } \mathrm{L}^2_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)\ ,$$

with $\rho, \mathbf{A}$ *in* $\mathrm{X}^1_{loc} \cap \mathrm{C}^2$, $\rho > 0$ *and* $\mathbf{A}$ *real positive definite (or replace it with its symmetric part, and subsume the lower order terms in the last term), and* S^k *be standard operators with symbols* s^k.
If $u^n \rightharpoonup 0$ *in* $\mathrm{H}^1_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)$, *then* ∇u^n *corresponds to an H-measure* $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\pi$, *then*

$$Q\pi = 0\ ,$$

and

$$\Big\langle \pi, \{\Psi, Q\} + (\xi_k s^k + \xi_k \bar{s}^k)\Psi \Big\rangle = 0\ ,$$

for $\Psi \in \mathrm{C}^1_c(\langle 0,T\rangle \times \Omega \times \mathrm{S}^d)$. ∎

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w \;. \end{cases}$$

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w \ . \end{cases}$$

We have used D'Alembert's formula for solution, our approach and the approach of P. Gérad, obtaining the same result in this special case (which is treatable by both methods, and explicit calculations).

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w \;. \end{cases}$$

We have used D'Alembert's formula for solution, our approach and the approach of P. Gérad, obtaining the same result in this special case (which is treatable by both methods, and explicit calculations).

Physically important quantity is *energy density:*

$$d(t,x) := \frac{1}{2}(u_t^2 + u_x^2) \;,$$

as well as the *energy* at time t: $$e(t) := \int_{\mathbf{R}} d(t,x)\, dx.$$

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w \ . \end{cases}$$

We have used D'Alembert's formula for solution, our approach and the approach of P. Gérad, obtaining the same result in this special case (which is treatable by both methods, and explicit calculations).

Physically important quantity is *energy density:*

$$d(t,x) := \frac{1}{2}(u_t^2 + u_x^2) \ ,$$

as well as the *energy* at time t: $\qquad e(t) := \int_{\mathbf{R}} d(t,x)\, dx.$

After simple calculations we get

$$4d(t,x) = \Big(v'(x+t) + w(x+t)\Big)^2 + \Big(v'(x-t) - w(x-t)\Big)^2 \ .$$

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w \,. \end{cases}$$

We have used D'Alembert's formula for solution, our approach and the approach of P. Gérad, obtaining the same result in this special case (which is treatable by both methods, and explicit calculations).

Physically important quantity is *energy density:*

$$d(t,x) := \frac{1}{2}(u_t^2 + u_x^2) \,,$$

as well as the *energy* at time t: $\qquad e(t) := \int_{\mathbf{R}} d(t,x)\,dx.$

After simple calculations we get

$$4d(t,x) = \Big(v'(x+t) + w(x+t)\Big)^2 + \Big(v'(x-t) - w(x-t)\Big)^2 \,.$$

Assume that the physical system is modelled by the above wave equation on the microscale. In order to pass to the macroscale, in the spirit of Tatar's programme, we have to pass to the weak limit.

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightharpoonup{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightharpoonup{\mathrm{L}^2(\mathbf{R})} 0 \; .$$

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightarrow{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightarrow{\mathrm{L}^2(\mathbf{R})} 0 \; .$$

It follows that

$$u_n \longrightarrow 0 \; ,$$

but $d_n \longrightarrow d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightarrow{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightarrow{\mathrm{L}^2(\mathbf{R})} 0 \, .$$

It follows that

$$u_n \longrightarrow 0 \, ,$$

but $d_n \longrightarrow d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Applying the div-rot lemma we arrive at *equipartition of energy*, i.e.

$u_t^2 - u_x^2 \longrightarrow 0$;

the kinetic and potential energy are balanced at the macroscopic level.

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightarrow{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightarrow{\mathrm{L}^2(\mathbf{R})} 0 \ .$$

It follows that

$$u_n \longrightarrow 0 \ ,$$

but $d_n \longrightarrow d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Applying the div-rot lemma we arrive at *equipartition of energy*, i.e.
$u_t^2 - u_x^2 \longrightarrow 0$;
the kinetic and potential energy are balanced at the macroscopic level.

In order to determine the solution completely, let us take periodically modulated initial conditions (we work in spaces $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R})$ and $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$):

$$v_n(x) := \frac{1}{n}\sin(nx) \qquad \text{and} \qquad w_n(x) := \sin(nx) \ .$$

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightharpoonup{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightharpoonup{\mathrm{L}^2(\mathbf{R})} 0 \, .$$

It follows that

$$u_n \longrightarrow 0 \, ,$$

but $d_n \longrightharpoonup d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Applying the div-rot lemma we arrive at *equipartition of energy*, i.e.
$u_t^2 - u_x^2 \longrightharpoonup 0$;
the kinetic and potential energy are balanced at the macroscopic level.

In order to determine the solution completely, let us take periodically modulated initial conditions (we work in spaces $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R})$ and $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$):

$$v_n(x) := \frac{1}{n}\sin(nx) \qquad \text{and} \qquad w_n(x) := \sin(nx) \, .$$

Simple calculations lead us to: $d_n(t,x) = 1 + \cos 2nx \sin 2nt \longrightharpoonup 1$,
weak $*$ in the space of Radon measures, therefore in the space of distributions as well.

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightarrow{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightarrow{\mathrm{L}^2(\mathbf{R})} 0 \ .$$

It follows that

$$u_n \longrightarrow 0 \ ,$$

but $d_n \longrightarrow d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Applying the div-rot lemma we arrive at *equipartition of energy*, i.e.
$u_t^2 - u_x^2 \longrightarrow 0$;
the kinetic and potential energy are balanced at the macroscopic level.

In order to determine the solution completely, let us take periodically modulated initial conditions (we work in spaces $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R})$ and $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$):

$$v_n(x) := \frac{1}{n}\sin(nx) \qquad \text{and} \qquad w_n(x) := \sin(nx) \ .$$

Simple calculations lead us to: $d_n(t,x) = 1 + \cos 2nx \sin 2nt \longrightarrow 1$, weak $*$ in the space of Radon measures, therefore in the space of distributions as well.

Even though the sequence of solutions (u_n) weakly converges to zero, the energy density is 1, equally distributed to kinetic and potential energy.

How this can be computed in general?

Two interesting quadratic forms:

$$q(x;\mathsf{v}) := \frac{1}{2}[\rho(x)v_0^2 + \mathbf{A}(x)v\cdot v]\ ,$$
$$Q(x;\mathsf{v}) := \frac{1}{2}[\rho(x)v_0^2 - \mathbf{A}(x)v\cdot v]\ .$$

Convergence of initial data and uniformly compact support imply:

$$u_n \xrightharpoonup{*} 0 \quad \text{in} \quad \mathrm{L}^\infty(\mathbf{R};\mathrm{H}^1)\cap \mathrm{W}^{1,\infty}(\mathbf{R};\mathrm{L}^2).$$

How this can be computed in general?

Two interesting quadratic forms:

$$q(x;\mathsf{v}) := \frac{1}{2}[\rho(x)v_0^2 + \mathbf{A}(x)v\cdot v]\ ,$$
$$Q(x;\mathsf{v}) := \frac{1}{2}[\rho(x)v_0^2 - \mathbf{A}(x)v\cdot v]\ .$$

Convergence of initial data and uniformly compact support imply:

$$u_n \xrightharpoonup{*} 0 \quad \text{in} \quad \mathrm{L}^\infty(\mathbf{R};\mathrm{H}^1)\cap \mathrm{W}^{1,\infty}(\mathbf{R};\mathrm{L}^2).$$

The energy density is $d_n = q(\nabla u_n)$.
Goal: compute the distributional limit d_n, i.e. the limit

$$D_n = \int_{\langle 0,T\rangle\times\mathbf{R}^d} d_n\phi\, dtdx\ .$$

How this can be computed in general?

Two interesting quadratic forms:

$$q(x;\mathsf{v}) := \frac{1}{2}[\rho(x)v_0^2 + \mathbf{A}(x)v\cdot v] \ ,$$
$$Q(x;\mathsf{v}) := \frac{1}{2}[\rho(x)v_0^2 - \mathbf{A}(x)v\cdot v] \ .$$

Convergence of initial data and uniformly compact support imply:

$$u_n \xrightharpoonup{*} 0 \quad \text{in} \quad \mathrm{L}^\infty(\mathbf{R};\mathrm{H}^1)\cap \mathrm{W}^{1,\infty}(\mathbf{R};\mathrm{L}^2).$$

The energy density is $d_n = q(\nabla u_n)$.
Goal: compute the distributional limit d_n, i.e. the limit

$$D_n = \int_{\langle 0,T\rangle\times\mathbf{R}^d} d_n\phi\, dtdx \ .$$

Results:

— Gilles Francfort & François Murat (1992): in linear case, C^∞ coefficients
— Patrick Gérard (1996): for constant coefficients, nonlinearity with term $u^p, p\leqslant 5$
— N. A. & Martin Lazar (2002): for symmetric hyperbolic systems

How this can be computed in general?

Two interesting quadratic forms:

$$q(x;\mathsf{v}) := \frac{1}{2}[\rho(x)v_0^2 + \mathbf{A}(x)v\cdot v] \; ,$$
$$Q(x;\mathsf{v}) := \frac{1}{2}[\rho(x)v_0^2 - \mathbf{A}(x)v\cdot v] \; .$$

Convergence of initial data and uniformly compact support imply:

$$u_n \xrightharpoonup{*} 0 \quad \text{in} \quad \mathrm{L}^\infty(\mathbf{R};\mathrm{H}^1)\cap \mathrm{W}^{1,\infty}(\mathbf{R};\mathrm{L}^2).$$

The energy density is $d_n = q(\nabla u_n)$.
Goal: compute the distributional limit d_n, i.e. the limit

$$D_n = \int_{\langle 0,T\rangle\times\mathbf{R}^d} d_n \phi \, dt dx \; .$$

Results:

— Gilles Francfort & François Murat (1992): in linear case, C^∞ coefficients
— Patrick Gérard (1996): for constant coefficients, nonlinearity with term $u^p, p\leqslant 5$
— N. A. & Martin Lazar (2002): for symmetric hyperbolic systems

We have attempted to do the same for semilinear wave equation ($d=3, p=3$), with variable coefficients. The difficulties led to the study of mixed-norm Lebesgue spaces, and also prompted the introduction of H-distributions.
For nonlinear equations L^2 theory usually does not work; one should try the L^p spaces.

Thank you for your attention!