H-measures and propagation of microlocal energy density

Nenad Antonić

Department of Mathematics
Faculty of Science
University of Zagreb

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Existence of H-measures

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^r)$, *then there exists its subsequence and a complex matrix Radon measure distribution of order zero* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and* $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$ *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 \mathsf{u}_{n'}} \otimes \widehat{\varphi_2 \mathsf{u}_{n'}} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)\, d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, (\varphi_1\bar{\varphi}_2) \boxtimes \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x})\bar{\varphi}_2(\mathbf{x})\psi(\boldsymbol{\xi})\, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi})\,.$$

∎

There are some other variants: (ultra)parabolic, fractional, one-scale, ...

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^d)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^d)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier $\mathcal{A}_a$, for $a \in \mathrm{L}^\infty(\mathbf{R}^d)$: $\widehat{\mathcal{A}_a u} = a\hat{u}$.
The norm is again equal to $\|a\|_{\mathrm{L}^\infty(\mathbf{R}^d)}$.

Delicate part: a is given only on S^{d-1}.
We extend it by the projection p: if α is a function defined on a compact surface, we take $a := \alpha \circ p$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\Big)$$

The precise scaling is contained in the projections, not the surface.

First commutation lemma

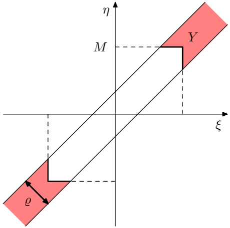
Lemma. (general form of the first commutation lemma — Luc Tartar)
If $b \in \mathrm{C}_0(\mathbf{R}^d)$ *and* $a \in \mathrm{L}^\infty(\mathbf{R}^d)$ *satisfy the condition*

$$(\forall\, \rho, \varepsilon \in \mathbf{R}^+)(\exists\, M \in \mathbf{R}^+) \quad |a(\boldsymbol{\xi}) - a(\boldsymbol{\eta})| \leqslant \varepsilon \;\; (\text{a.e. } (\boldsymbol{\xi}, \boldsymbol{\eta}) \in Y(M, \rho)) \;,$$

then $C := [\mathcal{A}_a, M_b]$ *is a compact operator on* $\mathrm{L}^2(\mathbf{R}^d)$. ∎

For given $M, \rho \in \mathbf{R}^+$ denote the set

$$Y = Y(M, \rho) = \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbf{R}^{2d} : |\boldsymbol{\xi}|, |\boldsymbol{\eta}| \geqslant M \;\&\; |\boldsymbol{\xi} - \boldsymbol{\eta}| \leqslant \rho\} \;.$$



[Some improvements in N.A., M. Mišur, D. Mitrović (2018)]

The importance of First commutation lemma

If we take $\mathsf{u}_n = (u_n, v_n)$, and consider $\mu = \mu_{12}$, we have

$$\begin{aligned}
\lim_{n'} \int_{\mathbf{R}^d} \widehat{\varphi_1 u_{n'}} \overline{\widehat{\varphi_2 v_{n'}}} \psi \, d\boldsymbol{\xi} &= \lim_{n'} \langle \mathcal{A}_\psi(\varphi_1 u_{n'}) | \varphi_2 v_{n'} \rangle \\
&= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(\varphi_1 u_{n'}) \overline{\varphi_2 v_{n'}} \, d\mathbf{x} \\
&= \lim_{n'} \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} \, d\mathbf{x} = \langle \mu, (\varphi_1 \bar{\varphi}_2) \boxtimes \psi \rangle \,.
\end{aligned}$$

Thus the limit is a bilinear functional in $\varphi_1 \bar{\varphi}_2$ and ψ, and we have the bound:

$$\Big| \int_{\mathbf{R}^d} \mathcal{A}_\psi(u_{n'}) \varphi_1 \overline{\varphi_2 v_{n'}} d\mathbf{x} \Big| \leqslant C \|\psi\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|\varphi_1 \overline{\varphi_2}\|_{\mathrm{C}_0(\mathbf{R}^d)} \,.$$

This form makes sense even for $p < 2$ (for $p > 2$ we use the fact that $u_n \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)$).

A class of symbols (L. Tartar)

Actually, we can consider more general operators than $\mathcal{A}_a$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m \alpha_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}(\mathrm{S}^{d-1})} \|b_m\|_{\mathrm{C}_0(\mathbf{R}^d)} = k < \infty$.

To such a symbol s, a *standard operator* $S_s \in \mathrm{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$ is assigned by

$$S_s = \sum_m \mathcal{A}_{a_m} M_b \ ,$$

with $\|S_s\|_{\mathrm{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$.

Clearly, S_s does not depend on the above decomposition, as

$$\widehat{S_s u}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} s(\mathbf{x}, \boldsymbol{\xi}/|\boldsymbol{\xi}|) u(\mathbf{x}) \, d\mathbf{x} \ ,$$

for u in a dense set of $\mathrm{L}^2(\mathbf{R}^d)$ (e.g. for $u \in \mathcal{S}$).

A class of symbols (cont.)

Any operator $A \in \mathrm{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$, which differs from S_s only by a compact operator, is an *operator of symbol* s, like

$$L_s = \sum_m M_{b_m} \mathcal{A}_{a_m} \ ,$$

where $\|L_s\|_{\mathrm{L}(\mathrm{L}^2(\mathbf{R}^d);\mathrm{L}^2(\mathbf{R}^d))} \leqslant k$. Neither L_s depends on the decomposition.

Theorem. *If* $\mathsf{u}_n \longrightarrow 0$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^r)$*, then there exists its subsequence and an H-measure* $\boldsymbol{\mu}$*, which is a Hermitian non-negative* $r \times r$ *matrix of distributions of order zero on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that for any* $\varphi_1, \varphi_2 \in \mathrm{C}_c(\mathbf{R}^d)$ *and any operators* $L_{s_1}, L_{s_2} \in \mathrm{L}(\mathrm{L}^2(\mathbf{R}^d); \mathrm{L}^2(\mathbf{R}^d))$*, with symbols* s_1, s_2 *one has*

$$\lim_{n'} \int_{\mathbf{R}^d} L_{s_1}(\varphi_1 u^j_{n'}) \overline{L_{s_2}(\varphi_2 u^k_{n'})} \, d\boldsymbol{\xi} = \langle \mu^{jk}, \varphi_1 s_1 \overline{\varphi_2 s_2} \rangle \ .$$

∎

We can consider $\Omega \subseteq \mathbf{R}^d$ as a domain, or even a manifold (with a volume form).
P. Gérard used a different approach, by using classical symbols. However, it is important to have symbols of lower regularity, as they come in applications from coefficients in PDEs.

▸ Classical symbols

Symmetric systems

$$\sum_k \mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \ , \ \mathbf{A}^k \text{ Hermitian}$$

Assume:

$$\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0 \quad \text{(weakly)},$$

$$\mathsf{f}^n \xrightarrow{\mathrm{H}^{-1}_{\mathrm{loc}}} 0 \quad \text{(strongly)}.$$

If supports of $\mathsf{u}^n, \mathsf{f}^n$ are contained inside Ω, we can extend them by zero to $\mathbf{R}^d$.

Theorem. (localisation property) *If* $\mathsf{u}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\mathbf{R}^d)^r$ *defines* $\boldsymbol{\mu}$*, and if* u^n *satisfies:*

$$\sum_k \partial_k \left(\mathbf{A}^k \mathsf{u}^n\right) \to 0 \ \text{ in the space } \mathrm{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^d)^r \ ,$$

then for $\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) := \sum_k \xi_k \mathbf{A}^k(\mathbf{x})$ *on* $\Omega \times S^{d-1}$ *it holds:*

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) \boldsymbol{\mu}^\top = \mathbf{0} \ .$$

∎

Thus, the support of H-measure $\boldsymbol{\mu}$ is contained in the set $\left\{(\mathbf{x}, \boldsymbol{\xi}) \in \Omega \times S^{d-1} : \det \mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = 0\right\}$ of points where $\mathbf{P}$ is a singular matrix.)

Example: scalar case

Consider $u^n \longrightarrow 0$ in $\mathrm{L}^2_{\mathrm{loc}}(\Omega)$ (after eventually passing to a subsequence)
it defines an H-measure μ
(a Radon measure: distribution of order zero and non-negative)

Assume that u^n are solutions of a PDE, or even less, that for a $\mathsf{b} \in \mathrm{C}^1(\Omega; \mathbf{R}^d)$

$$\mathsf{b} \cdot \nabla u^n \longrightarrow 0 \qquad \text{strongly in } \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega) \,,$$

then

$$P\mu = 0 \qquad \text{on } \Omega \times \mathrm{S}^{d-1} \,,$$

where $P(\mathbf{x}, \boldsymbol{\xi}) := \boldsymbol{\xi} \cdot \mathsf{b}(\mathbf{x})$.

The support of μ is contained within the zero set of P.

In particular, when $P \neq 0$ on $\Omega \times \mathrm{S}^{d-1}$, then $\mu = 0$
(and we have that $u^n \longrightarrow 0$ strongly).

Example: gradients

$\mathsf{u}^n \rightharpoonup 0$ weakly in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbf{R}^d)$, and defines $\boldsymbol{\mu}$.
Additionally assume:

$$\mathrm{rot}\, \mathsf{u}^n \longrightarrow \mathbf{O} \qquad \text{strongly in } \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega; \mathrm{L}(\mathbf{R}^d; \mathbf{R}^d)) \; .$$

Then there is a scalar non-negative Radon measure π on $\Omega \times \mathrm{S}^{d-1}$ such that

$$\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\pi \; .$$

The above extends to gradients of vectors as well.
$\mathsf{u}^n \rightharpoonup 0$ weakly in $\mathrm{H}^1_{\mathrm{loc}}(\Omega; \mathbf{R}^r)$
$\mathbf{V}^n := \nabla \mathsf{u}^n$ (the values are in $\mathrm{M}_{r\times d}$)

Then $\mathbf{V}^n \rightharpoonup \mathbf{O}$ and defines (possibly on a subsequence) H-measure $\boldsymbol{\mu}$ (*values* in $\mathrm{M}_{r\times d\times r\times d}$).

Then there is a $r \times r$ non-negative Hermitian symmetric Radon measure $\boldsymbol{\pi}$ on $\Omega \times \mathrm{S}^{d-1}$ such that

$$\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\boldsymbol{\pi} \; .$$

Example: linearised elasticity

Take $d \geqslant 2$, $\Omega \subseteq \mathbf{R}^d$ open, $\Omega_T := \langle 0, T\rangle \times \Omega$; $t = x_0$.

$\mathsf{u}^n \longrightarrow 0$ in $\mathrm{H}^1_{\mathrm{loc}}(\Omega_T)$, then $\mathbf{V}^n := \nabla \mathsf{u}^n$ defines an H-measure

$$\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \otimes \boldsymbol{\pi} \ .$$

Additionally assume that u^n satisfies a *Cauchy-Lamé-like* system

$$\Big(\rho(\mathsf{u}^n)'\Big)' - \operatorname{div} \boldsymbol{\sigma} \longrightarrow 0 \qquad \text{in } \mathrm{H}^{-1}_{\mathrm{loc}}(\Omega_T) \ ,$$

where

$$\boldsymbol{\sigma}^n = \mathsf{C}[\mathbf{E}] \ , \quad \mathbf{E} = \mathrm{Sym}\nabla \mathsf{u}^n \ ,$$

while ρ and C are continuous functions on Ω_T. [C takes symmetric matrices to symmetric matrices, and is symmetric]

Then $\boldsymbol{\pi}$ satisfies

$$\Big(\tau^2 \rho \mathbf{I} - \mathbf{A}(\boldsymbol{\xi}')\Big)\boldsymbol{\pi} = \mathbf{O} \ ,$$

where the *acoustic tensor* is

$$\mathbf{A}(\boldsymbol{\xi}') = \mathsf{C}(\boldsymbol{\xi}' \otimes \boldsymbol{\xi}') \ .$$

The $\operatorname{supp} \boldsymbol{\pi}$ is contained in the set where $\det \Big(\tau^2 \rho \mathbf{I} - \mathbf{A}(\boldsymbol{\xi}')\Big) = 0$.

Second commutation lemma

$$X^m := \Big\{ w \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) : (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^H d)\ |\boldsymbol{\alpha}| \leqslant m \Longrightarrow w^{(\boldsymbol{\alpha})} \in \mathcal{F}(\mathrm{L}^1(\mathbf{R}^d)) \Big\}$$

is a Banach space with norm

$$\|w\|_{X^m} := \int_{\mathbf{R}^d} \big(1 + 4\pi^2 |\boldsymbol{\xi}|^2\big)^{m/2} \big|\hat{w}(\boldsymbol{\xi})\big|\, d\boldsymbol{\xi}\ .$$

$X^m \subseteq \mathrm{C}^m(\mathbf{R}^d)$, and the derivatives up to order m vanish at infinity (they are in $\mathrm{C}_0(\mathbf{R}^d)$).
On the other hand, $\mathrm{H}^s(\mathbf{R}^d) \subseteq X^m$, for $s > m + \frac{d}{2}$.
X^m is an algebra with respect to the multiplication of functions; it holds:

$$\|f * g\|_{\mathrm{L}^1} \leqslant \|f\|_{\mathrm{L}^1} \|g\|_{\mathrm{L}^1}$$
$$\|\hat{f} \cdot \hat{g}\|_{X^0} \leqslant \|\hat{f}\|_{X^0} \|\hat{g}\|_{X^0}$$

$X^m_{\mathrm{loc}}(\Omega)$: the space of all functions u such that $\varphi u \in X^m$, for $\varphi \in \mathrm{C}_c^\infty(\Omega)$.

Lemma. *Let $\mathcal{A}_a$, M_b be standard operators, with symbols a, b, such that $\alpha \in \mathrm{C}^1(S^{d-1})$ and $b \in X^1$.*
Then $C := [\mathcal{A}_a, M_b] \in \mathcal{L}\big(\mathrm{L}^2(\mathbf{R}^d), \mathrm{H}^1(\mathbf{R}^d)\big)$, and ∇C has a symbol $(\nabla_{\boldsymbol{\xi}} a \cdot \nabla_{\mathbf{x}} b)\boldsymbol{\xi}$.
(we extend α to a homogeneous function a on $\mathbf{R}^d_* := \mathbf{R}^d \setminus \{\mathbf{0}\}$) ∎

A smaller class of symbols (L. Tartar)

Corollary. *Under the above assumptions,*

$$\mathcal{A}_a M_b \partial_j u = M_b \partial_j (\mathcal{A}_a u) + Lu, \quad u \in \mathrm{L}^2(\mathbf{R}^d) \ ,$$

where L has a symbol $\xi_j\{a, b\}$. ∎

Actually, we can consider more general operators than $\mathcal{A}_a$ and M_b. We can consider the *symbols* of the form

$$s(\mathbf{x}, \boldsymbol{\xi}) = \sum_m a_m(\boldsymbol{\xi}) b_m(\mathbf{x}) \ ,$$

with $\sum_m \|\alpha_m\|_{\mathrm{C}^1(\mathrm{S}^{d-1})} \|b_m\|_{X^1} < \infty$,
and *standard operators* $S_s = \sum_m \mathcal{A}_{a_m} M_b$.

Lemma. *If S_1, S_2 are standard operators with symbols s_1, s_2 as above, then*

$$\frac{\partial}{\partial x^j}[S_1, S_2] \quad \textit{has symbol} \quad \xi_j\{s_1, s_2\} \ .$$

∎

The Poisson bracket is $\{p, q\} := \nabla_{\boldsymbol{\xi}} p \cdot \nabla_{\mathbf{x}} q - \nabla_{\mathbf{x}} p \cdot \nabla_{\boldsymbol{\xi}} q$.

Propagation property for symmetric systems

$$\mathbf{A}^k \partial_k \mathsf{u} + \mathbf{B}\mathsf{u} = \mathsf{f} \quad , \qquad \mathbf{A}^k \text{ Hermitian}$$

Theorem. *Let* $\mathbf{A}^k \in \mathrm{C}_0^1(\Omega; \mathrm{M}_{r\times r})$.
If $(\mathsf{u}^n, \mathsf{f}^n)$ *satisfy the above for* $n \in \mathbf{N}$, *and* $\mathsf{u}^n, \mathsf{f}^n \longrightarrow 0$ *in* $\mathrm{L}^2(\Omega)$,
then for any $\psi \in \mathrm{C}_0^1(\Omega \times S^{d-1})$, *the H-measure associated to sequence* $(\mathsf{u}^n, \mathsf{f}^n)$*:*

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \boldsymbol{\mu}_{12} \\ \boldsymbol{\mu}_{21} & \boldsymbol{\mu}_{22} \end{bmatrix},$$

satisfies:

$$\left\langle \boldsymbol{\mu}_{11}, \{\mathbf{P}, \psi\} + \psi \partial_k \mathbf{A}^k - 2\psi \mathbf{S} \right\rangle + \langle 2\mathrm{Re}\,\mathrm{tr}\boldsymbol{\mu}_{12}, \psi \rangle = 0 \;,$$

where $\mathbf{S} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^*)$. *[Recall:* $\mathbf{P} = \xi_k \mathbf{A}^k$ *]* ∎

$\boldsymbol{\mu}$ is associated to the pair of sequences $(\mathsf{u}^n, \mathsf{f}^n)$, the block $\boldsymbol{\mu}_{11}$ is determined by u^n, $\boldsymbol{\mu}_{22}$ with f^n, while the non-diagonal blocks correspond to the product of u^n and f^n.

The equation for H-measure

Corollary. *In the sense of distributions on* $\Omega \times S^{d-1}$ *the H-measure* $\boldsymbol{\mu}$ *satisfies:*

$$\partial^l \mathbf{P} \cdot \partial_l \boldsymbol{\mu}_{11} - \partial^l_t (\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) + (d-1)(\partial_l \mathbf{P} \cdot \boldsymbol{\mu}_{11}) \xi^l \\ + (2\mathbf{S} - \partial_l \mathbf{A}^l) \cdot \boldsymbol{\mu}_{11} = 2 \mathrm{Re}\, \mathrm{tr} \boldsymbol{\mu}_{12} \,,$$

where $\partial^l_t := \partial^l - \xi^l \xi_k \partial^k$ *is the tangential gradient on the unit sphere.* ∎

This allows us to investigate the behaviour of H-measures as solutions of initial-value problems, with appropriate initial conditions.
Besides the wave equations, there are applications to Maxwell's and Dirac's systems, even to the equations that change their type (like the Tricomi equation).

The wave equation

$$(\rho u')' - \mathsf{div}\,(\mathbf{A}\nabla u) = g\ .$$

It can be written as an equivalent symmetric system ($t = x^0$ and $\partial_0 := \frac{\partial}{\partial t}$):

$$\partial_0(\rho\partial_0 u) - \sum_{i,j=1}^{d} \partial_i(a^{ij}\partial_j u) = g\ .$$

By introducing: $v_j := \partial_j u$, for $j \in 0..d$, we obtain (Schwarz' symmetries!):

$$\begin{bmatrix} \rho & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \mathbf{A} & \\ 0 & & & \end{bmatrix} \partial_0 \mathsf{v} + \sum_{i=1}^{d} \begin{bmatrix} 0 & -a^{i1} & \cdots & -a^{id} \\ -a^{i1} & & & \\ \vdots & & \mathbf{0} & \\ -a^{id} & & & \end{bmatrix} \partial_i \mathsf{v}$$

$$+ \begin{bmatrix} b^0 & b^1 & \cdots & b^d \\ 0 & & & \\ \vdots & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \mathsf{v} = \begin{bmatrix} g \\ 0 \\ \vdots \\ 0 \end{bmatrix}\ .$$

The symbol of differential operator is:

$$\mathbf{P}(\mathbf{x}, \boldsymbol{\xi}) = \xi_k \mathbf{A}^k(\mathbf{x}) = \begin{bmatrix} \xi_0\rho & -(\mathbf{A}\boldsymbol{\xi}')^\top \\ -\mathbf{A}\boldsymbol{\xi}' & \xi_0\mathbf{A} \end{bmatrix}\ .$$

Transport of H-measures associated to the wave equation

From the localisation property we can conclude that $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\nu$. For the right hand side of the equation we have:

$$\langle \gamma, \varphi_1 \bar{\varphi}_2 \psi \rangle := \lim_n \int_{\mathbf{R}^{d+1}} \widehat{\varphi_1 v_{0,n}}(\boldsymbol{\xi}) \overline{\widehat{\varphi_2 g_n}(\boldsymbol{\xi})} \psi \left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right) d\boldsymbol{\xi} \; .$$

Theorem. *On $\mathbf{R}^{d+1} \times S^d$ measure ν satisfies ($Q := \rho\xi_0^2 - \mathbf{A}\boldsymbol{\xi}' \cdot \boldsymbol{\xi}'$):*

$$\nabla_{\boldsymbol{\xi}} Q \cdot \nabla_{\mathbf{x}} (\xi_0 \nu) - Q \partial_0 \nu + \big(\boldsymbol{\xi} \otimes \boldsymbol{\xi} - \mathbf{I}\big) \nabla_{\mathbf{x}} Q \cdot \nabla_{\boldsymbol{\xi}} (\xi_0 \nu) + (d+2) \big(\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}\big)(\xi_0 \nu) = 2\mathrm{Re}\, \gamma$$

∎

The equation can be written in a nicer form:

$$\{Q, \xi_0 \nu\} + (\nabla_{\mathbf{x}} Q \cdot \boldsymbol{\xi}) \big[\boldsymbol{\xi} \cdot \nabla_{\boldsymbol{\xi}} (\xi_0 \nu) + (d+2)(\xi_0 \nu)\big] - Q \partial_0 \nu = 2\mathrm{Re}\, \gamma \; .$$

Recent Tartar's result (2017)

Theorem. *Let* $u^n \in \mathrm{C}^0(0,T;\mathrm{H}^1(\Omega)) \cap \mathrm{C}^1(0,T;\mathrm{L}^2(\Omega))$ *be a sequence of solutions of a "wave equation"*

$$(\rho(u^n)')' - \mathrm{div}\,(\mathbf{A}\nabla u^n) + S^k\partial_k u^n \longrightarrow 0 \qquad \textit{in } \mathrm{L}^2_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)\ ,$$

with $\rho, \mathbf{A}$ *in* $\mathrm{X}^1_{loc} \cap \mathrm{C}^2$, $\rho > 0$ *and* $\mathbf{A}$ *real positive definite (or replace it with its symmetric part, and subsume the lower order terms in the last term), and* S^k *be standard operators with symbols* s^k.
If $u^n \longrightarrow 0$ *in* $\mathrm{H}^1_{\mathrm{loc}}(\langle 0,T\rangle \times \Omega)$, *then* ∇u^n *corresponds to an H-measure* $\boldsymbol{\mu} = (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\pi$, *then*

$$Q\pi = 0\ ,$$

and

$$\Big\langle \pi, \{\Psi, Q\} + (\xi_k s^k + \xi_k \bar{s}^k)\Psi \Big\rangle = 0\ ,$$

for $\Psi \in \mathrm{C}^1_c(\langle 0,T\rangle \times \Omega \times \mathrm{S}^d)$. ∎

An explicit example

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(0,\cdot) = v \\ u_t(0,\cdot) = w \ . \end{cases}$$

We have used D'Alembert's formula for solution, our approach and the approach of P. Gérad, obtaining the same result in this special case (which is treatable by both methods, and explicit calculations).

Physically important quantity is *energy density:*

$$d(t,x) := \frac{1}{2}(u_t^2 + u_x^2) \ ,$$

as well as the *energy* at time t: $\qquad e(t) := \int_{\mathbf{R}} d(t,x)\, dx.$

After simple calculations we get

$$4d(t,x) = \Big(v'(x+t) + w(x+t)\Big)^2 + \Big(v'(x-t) - w(x-t)\Big)^2 \ .$$

Assume that the physical system is modelled by the above wave equation on the microscale. In order to pass to the macroscale, in the spirit of Tatar's programme, we have to pass to the weak limit.

Oscillating initial data

Let (v_n) and (w_n) be sequences of initial data, determining the sequence of solutions (u_n), such that:

$$v_n \xrightarrow{\mathrm{H}^1(\mathbf{R})} 0 \qquad \text{and} \qquad w_n \xrightarrow{\mathrm{L}^2(\mathbf{R})} 0 \, .$$

It follows that

$$u_n \longrightarrow 0 \, ,$$

but $d_n \longrightarrow d \geqslant 0$ weakly $*$ in the space of Radon measures; in general d is not zero.

Applying the div-rot lemma we arrive at *equipartition of energy*, i.e.
$u_t^2 - u_x^2 \longrightarrow 0$;
the kinetic and potential energy are balanced at the macroscopic level.

In order to determine the solution completely, let us take periodically modulated initial conditions (we work in spaces $\mathrm{H}^1_{\mathrm{loc}}(\mathbf{R})$ and $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$):

$$v_n(x) := \frac{1}{n}\sin(nx) \qquad \text{and} \qquad w_n(x) := \sin(nx) \, .$$

Simple calculations lead us to: $d_n(t,x) = 1 + \cos 2nx \sin 2nt \longrightarrow 1$, weak $*$ in the space of Radon measures, therefore in the space of distributions as well.

Even though the sequence of solutions (u_n) weakly converges to zero, the energy density is 1, equally distributed to kinetic and potential energy.

How this can be computed in general?

Two interesting quadratic forms:

$$q(x;\mathsf{v}) := \frac{1}{2}\left[\rho(x)v_0^2 + \mathbf{A}(x)v\cdot v\right] ,$$
$$Q(x;\mathsf{v}) := \frac{1}{2}\left[\rho(x)v_0^2 - \mathbf{A}(x)v\cdot v\right] .$$

Convergence of initial data and uniformly compact support imply:

$$u_n \xrightharpoonup{*} 0 \quad \text{in} \quad \mathrm{L}^\infty(\mathbf{R};\mathrm{H}^1)\cap \mathrm{W}^{1,\infty}(\mathbf{R};\mathrm{L}^2).$$

The energy density is $d_n = q(\nabla u_n)$.
Goal: compute the distributional limit d_n, i.e. the limit

$$D_n = \int_{\langle 0,T\rangle\times\mathbf{R}^d} d_n\phi\, dtdx \,.$$

Results:

— Gilles Francfort & François Murat (1992): in linear case, C^∞ coefficients

— Patrick Gérard (1996): for constant coefficients, nonlinearity with term $u^p, p \leqslant 5$

— N. A. & Martin Lazar (2002): for symmetric hyperbolic systems

We have attempted to do the same for semilinear wave equation $(d=3, p=3)$, with variable coefficients. The difficulties led to the study of mixed-norm Lebesgue spaces, and also prompted the introduction of H-distributions.
For nonlinear equations L^2 theory usually does not work; one should try the L^p spaces.

A general view

We can unify the results: consider equations of the form

$$P_0(\varrho P_0 u_n) + \mathsf{P}_1 \cdot \mathbf{A}\mathsf{P}_1 u_n = 0\,,$$

where P_0 and P_1 stand for (pseudo)differential operators in time and space variables, with (principal) symbols p_0 and p_1, and $Q = \varrho p_0^2 + \mathbf{A}\mathsf{p}_1 \cdot \mathsf{p}_1$ being the symbol of the differential operator defining the left-hand side of the equation. For the H-measure $\tilde{\boldsymbol{\mu}}$ associated to $(P_0 u_n, \mathsf{P}_1 u_n)$, converging weakly in L^2 to 0, $\tilde{\boldsymbol{\mu}}$ is of the form

$$\tilde{\boldsymbol{\mu}} = \frac{\overline{\mathsf{p} \otimes \mathsf{p}}}{|\mathsf{p}|^2}\tilde{\nu}\,,$$

where $\tilde{\nu} := \mathrm{tr}\tilde{\boldsymbol{\mu}}$ is a scalar measure, and the localisation principle reads

$$Q\tilde{\nu} = 0\,.$$

Finally, the propagation principle states

$$\Big\langle \frac{\xi_m \tilde{\nu}}{|\mathsf{p}|^2}, \{\phi, Q\} \Big\rangle + \Big\langle \frac{\nu}{|\mathsf{p}|^2}, p\,\partial_m Q \Big\rangle = 0\,.$$

This covers both the classical and the parabolic case.

Thank you for your attention!

What are H-measures?

Mathematical objects introduced by:

- Luc Tartar, motivated by intended applications in homogenisation (H), and
- Patrick Gérard, whose motivation were certain problems in kinetic theory (and who called these objects *microlocal defect measures*).

Start from $u_n \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, $\varphi \in \mathrm{C}_c(\mathbf{R}^d)$, and take the Fourier transform:

$$\widehat{\varphi u_n}(\boldsymbol{\xi}) = \int_{\mathbf{R}^d} e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} (\varphi u_n)(\mathbf{x}) d\mathbf{x}$$

As φu_n is supported on a fixed compact set K, so $|\widehat{\varphi u_n}(\boldsymbol{\xi})| \leqslant C$.
Furthermore, $u_n \rightharpoonup 0$, and from the definition $\widehat{\varphi u_n}(\boldsymbol{\xi}) \longrightarrow 0$ pointwise.

By the Lebesgue dominated convergence theorem applied on bounded sets

$$\widehat{\varphi u_n} \longrightarrow 0 \text{ strong, i.e. strongly in } \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}^d)\ .$$

On the other hand, by the Plancherel theorem: $\|\widehat{\varphi u_n}\|_{\mathrm{L}^2(\mathbf{R}^d)} = \|\varphi u_n\|_{\mathrm{L}^2(\mathbf{R}^d)}$.

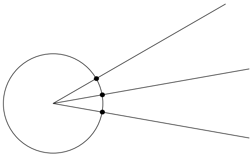
If $\varphi u_n \not\rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{R}^d)$, then $\widehat{\varphi u_n} \not\rightharpoonup 0$; some information must go to infinity.

Limit is a measure

How does it go to infinity in various directions? Take $\psi \in \mathrm{C}(\mathrm{S}^{d-1})$, and consider:

$$\lim_n \int_{\mathbf{R}^d} \psi(\boldsymbol{\xi}/|\boldsymbol{\xi}|)|\widehat{\varphi u_n}|^2 d\boldsymbol{\xi} = \int_{\mathrm{S}^{d-1}} \psi(\boldsymbol{\xi}) d\nu_\varphi(\boldsymbol{\xi}) \ .$$

The limit is a linear functional in ψ, thus an integral over the sphere of some nonnegative Radon measure (a bounded sequence of Radon measures has an accumulation point), which depends on φ. How does it depend on φ?



Theorem. (u^n) *a sequence in* $\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^r)$, $\mathsf{u}^n \xrightarrow{\mathrm{L}^2} 0$ *(weakly), then there is a subsequence* $(\mathsf{u}^{n'})$ *and* $\boldsymbol{\mu}$ *on* $\mathbf{R}^d \times \mathrm{S}^{d-1}$ *such that:*

$$\lim_{n'\to\infty} \int_{\mathbf{R}^d} \mathcal{F}\Big(\varphi_1 \mathsf{u}^{n'}\Big) \otimes \mathcal{F}\Big(\varphi_2 \mathsf{u}^{n'}\Big) \psi\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right) d\boldsymbol{\xi} = \langle \boldsymbol{\mu}, \varphi_1 \bar{\varphi}_2 \psi \rangle$$

$$= \int_{\mathbf{R}^d \times \mathrm{S}^{d-1}} \varphi_1(\mathbf{x}) \bar{\varphi}_2(\mathbf{x}) \psi(\boldsymbol{\xi}) \, d\bar{\boldsymbol{\mu}}(\mathbf{x}, \boldsymbol{\xi}) \ .$$

Why a parabolic variant?

Parabolic pde-s are:

- well studied, and we have good theory for them
- in some cases we even have explicit solutions (by formulae)

$1 : 2$ is certainly a good ratio to start with

Besides the immediate applications (which motivated this research), related to the properties of parabolic equations, applications are possible to other equations and problems involving the scaling $1 : 2$.
Naturally, after understanding this ratio $1 : 2$, other ratios should be considered as well, as required by intended applications.

Terminology: *classical* as opposed to *parabolic or variant* H-measures.
The sphere we replace by:

$$\sigma^4(\tau, \boldsymbol{\xi}) := (2\pi\tau)^2 + (2\pi|\boldsymbol{\xi}|)^4 = 1 \ , \ \text{or}$$

$$\sigma_1^2(\tau, \boldsymbol{\xi}) := |\tau| + (2\pi|\boldsymbol{\xi}|)^2 = 1 \ .$$

finally we chose the ellipse

$$\rho^2(\tau, \boldsymbol{\xi}) := |\boldsymbol{\xi}/2|^2 + \sqrt{(\boldsymbol{\xi}/2)^4 + \tau^2} = 1 \ .$$

Notation.
For simplicity (2D): $(t, x) = (x^0, x^1) = \mathbf{x}$ and $(\tau, \xi) = (\xi_0, \xi_1) = \boldsymbol{\xi}$.

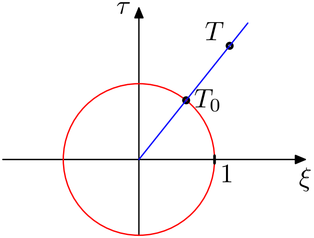
We use the Fourier transform in both space and time variables.

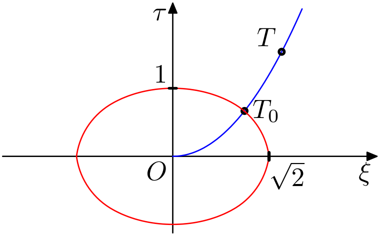
Rough geometric idea

Take a sequence $\mathrm{u}_n \longrightarrow 0$ in $\mathrm{L}^2(\mathbf{R}^2)$, and integrate $|\widehat{\varphi \mathrm{u}_n}|^2$ along

rays and project onto S^1

parabolas and project onto P^1





In $\mathbf{R}^2$ we have a compact curve (a surface in higher dimensions):

$$S^1 \ldots\ r^2(\tau,\xi) := \tau^2 + \xi^2 = 1 \qquad P^1 \ldots\ \rho^2(\tau,\xi) := (\xi/2)^2 + \sqrt{(\xi/2)^4 + \tau^2} = 1$$

and projection of $\mathbf{R}^2_* = \mathbf{R}^2 \setminus \{0\}$ onto the curve (surface):

$$p(\tau,\xi) := \left(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\right) \qquad \pi(\tau,\xi) := \left(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\right)$$

Analytic picture

Multiplication by $b \in \mathrm{L}^\infty(\mathbf{R}^2)$, a bounded operator M_b on $\mathrm{L}^2(\mathbf{R}^2)$:
$(M_b u)(\mathbf{x}) := b(\mathbf{x})u(\mathbf{x})$, norm equal to $\|b\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Fourier multiplier $\mathcal{A}_\alpha$, for $\alpha \in \mathrm{L}^\infty(\mathbf{R}^2)$: $\widehat{\mathcal{A}_\alpha u} = \alpha \hat{u}$.
The norm is again equal to $\|\alpha\|_{\mathrm{L}^\infty(\mathbf{R}^2)}$.

Delicate part: α is given only on S^1 or P^1.
We extend it by the projections, p or π: if α is a function defined on a compact surface, we take $a := \alpha \circ p$ or $a := \alpha \circ \pi$, i.e.

$$a(\tau, \xi) := \alpha\Big(\frac{\tau}{r(\tau,\xi)}, \frac{\xi}{r(\tau,\xi)}\Big) \qquad a(\tau, \xi) := \alpha\Big(\frac{\tau}{\rho^2(\tau,\xi)}, \frac{\xi}{\rho(\tau,\xi)}\Big)$$

The precise scaling is contained in the projections, not the surface.

Back

Classical (Hörmander) symbols

Let $k(\boldsymbol{\xi}) := \sqrt{1+4\pi^2|\boldsymbol{\xi}|^2}$; then for $m \in \mathbf{R}$, $\rho \in \langle 0,1]$ and $\delta \in [0,1\rangle$ we define $\mathrm{S}^m_{\rho,\delta}$ as the set:

$$\left\{a \in \mathrm{C}^\infty(\mathbf{R}^d \times \mathbf{R}^d) : (\forall\, \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbf{N}_0^d)(\exists\, C_{\boldsymbol{\alpha},\boldsymbol{\beta}} > 0) \quad \left|\partial_{\boldsymbol{\beta}}\partial^{\boldsymbol{\alpha}} a\right| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}} k^{m-\rho|\boldsymbol{\alpha}|+\delta|\boldsymbol{\beta}|}\right\} .$$

$\mathrm{S}^m_{\rho,\delta}$ is a vector space, and its elements are called *symbols* of order m and type ρ,δ. The best possible constant $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is taken as the value of corresponding seminorm of a in $\mathrm{S}^m_{\rho,\delta}$, and in such a way these seminorms make $\mathrm{S}^m_{\rho,\delta}$ into a Fréchet space. While $\mathrm{S}^m_{1,0}$ is the space of classical symbols (sometimes denoted only by S^m), for parabolic H-measures we have used $\mathrm{S}^m_{\frac{1}{2},0}$.

There are also local versions of these symbol classes.

The symbols are used to define corresponding operators on $\mathcal{S}$ (and by transposition, also on $\mathcal{S}'$) by:

$$a(\cdot;D)\varphi := \bar{\mathcal{F}}(a\hat{\varphi}) .$$

This is a generalisation of earlier considered operators M_ϕ and P_ψ.

For such operators and symbols, some calculus rules are valid; for example, if $a \in \mathrm{S}^k_{\frac{1}{2},0}$ and $b \in \mathrm{S}^m_{\frac{1}{2},0}$, then the symbol of composition operator $a(\cdot,D)b(\cdot,D)$ is in $\mathrm{S}^{k+m}_{\frac{1}{2},0}$, and it is given by asymptotic expansion

$$\sum_{|\boldsymbol{\alpha}|\geqslant 0} \frac{1}{\boldsymbol{\alpha}!}(\partial_{\boldsymbol{\alpha}} a)(D^{\boldsymbol{\alpha}} b) .$$