Small-amplitude homogenisation of Kirchhoff-Love plate

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Kirchhoff-Love plate theory

Homogenisation of Kirchhoff-Love plates

Small-amplitude homogenisation

Assumptions

- the plate is thin, but not very thin
  (rougly, the thickness is 1–20% of the leading dimension)
- the plate thickness might vary only slowly
  (so that the 3D stress effects are ignored)
- the plate is symmetric about mid-surface
- applied transverse loads are distributed over plate surface areas more than t^2 (no concentrated loads)
- there is no significant extension of the mid-surface

  There are no transverse shear deformations.
  The variation of vertical displacement in the direction of thickness can be neglected.
  The planes perpendicular to the mid-surface will remain plane and perpendicular to the deformed mid-surface.

Kirchhoff-Love plate equation

The above leads to a linear elliptic problem, with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}_0^2(\Omega)\,, \end{cases}$$

where:

- $\Omega \subseteq \mathbf{R}^d$ is a bounded domain ($d = 2$ ... for the plate)
- $f \in \mathrm{H}^{-2}(\Omega)$ is the external load
- $u \in \mathrm{H}_0^2(\Omega)$ is the vertical displacement of the plate
- **M** describes (non-homogeneous) properties of the material plate is made of; more precisely, **M** is taken from the set:

$$\mathfrak{M}_2(\alpha, \beta; \Omega) := \Big\{ \mathbf{N} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) : (\forall \mathbf{S} \in \mathrm{Sym}) \quad \mathbf{N}(\mathbf{x})\mathbf{S} : \mathbf{S} \geqslant \alpha \mathbf{S} : \mathbf{S} \ (\text{ae } \mathbf{x}) \ \& \ \mathbf{N}^{-1}(\mathbf{x})\mathbf{S} : \mathbf{S} \geqslant \frac{1}{\beta}\mathbf{S} : \mathbf{S} \ (\text{ae } \mathbf{x}) \Big\}$$

This ensures the boundedness and coercivity, so we have the existence and uniqueness of solutions via the Lax-Milgram lemma in a standard way.

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H-convergence

A sequence of tensor functions $(\mathbf{M}^n)$ in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ H-converges to $\mathbf{M} \in \mathfrak{M}_2(\alpha', \beta'; \Omega)$ if for any $f \in \mathrm{H}^{-2}(\Omega)$ the sequence of solutions u_n of problems

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

coverges weakly to a limit u in $\mathrm{H}^2_0(\Omega)$, while the sequence $(\mathbf{M}^n \nabla\nabla u_n)$ converges to $\mathbf{M}\nabla\nabla u$ weakly in the space $\mathrm{L}^2(\Omega; \mathrm{Sym})$.

general form for higher-order elliptic equations:
Žikov, Kozlov, Oleinik, Ngoan, 1979

for plates: N.A. & N. Balenović, 1999
revisited: K. Burazin & J. Jankov, 2019 (preprint)

Compactness

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*. Then there is a subsequence* $(\mathbf{M}^{n_k})$ *and a tensor function* $\mathbf{M} \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ *such that* $(\mathbf{M}^{n_k})$ *H-converges to* $\mathbf{M}$*.* ■

Theorem. *(compactness by compensation) Let the following convergences be valid:*

$$w^n \rightharpoonup w^\infty \quad \text{in} \quad \mathrm{H}^2_{\mathrm{loc}}(\Omega)\,,$$
$$\mathbf{D}^n \rightharpoonup \mathbf{D}^\infty \quad \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})\,,$$

with an additional assumption that the sequence $(\operatorname{div}\operatorname{div}\mathbf{D}^n)$ *is contained in a precompact (for the strong topology) set of the space* $\mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$*. Then we have*

$$\nabla\nabla w^n : \mathbf{D}^n \xrightharpoonup{*} \nabla\nabla w^\infty : \mathbf{D}^\infty$$

in the space of Radon measures. ■

Locality and irrelevance of boundary conditions

Theorem. *(locality of H-convergence) Let* (M^n) *and* (O^n) *be two sequences of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, which H-converge to* M *and* O*, respectively. Let* ω *be an open subset compactly embedded in* Ω*. If* $\mathsf{M}^n(\mathbf{x}) = \mathsf{O}^n(\mathbf{x})$ *in* ω*, then* $\mathsf{M}(\mathbf{x}) = \mathsf{O}(\mathbf{x})$ *in* ω*.* ∎

Theorem. *(irrelevance of boundary conditions) Let* (M^n) *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converges to* M*. For any sequence* (z_n) *such that*

$$\begin{aligned} z_n &\longrightarrow z \quad \text{in } \mathrm{H}^2_{\mathrm{loc}}(\Omega) \\ \operatorname{div}\operatorname{div}(\mathsf{M}^n \nabla\nabla z_n) = f_n &\longrightarrow f \quad \text{in } \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega), \end{aligned}$$

the weak convergence $\mathsf{M}^n \nabla\nabla z_n \rightharpoonup \mathsf{M}\nabla\nabla z$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathrm{Sym})$ *holds.* ∎

Convergence of energies

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converges to* $\mathbf{M}$*. For any* $f \in \mathrm{H}^{-2}(\Omega)$*, the sequence* (u_n) *of solutions of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in H_0^2(\Omega)\,. \end{cases}$$

satisfies $\mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \rightharpoonup \mathbf{M}\nabla\nabla u : \nabla\nabla u$ *weakly-$*$ in the space of Radon measures and* $\int_\Omega \mathbf{M}^n \nabla\nabla u_n : \nabla\nabla u_n \, d\mathbf{x} \to \int_\Omega \mathbf{M}\nabla\nabla u : \nabla\nabla u \, d\mathbf{x}$*, where* u *is the solution of the homogenised equation*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M} \nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}_0^2(\Omega)\,. \end{cases}$$

∎

Ordering property . . .

Theorem. *Let* $(\mathbf{M}^n)$ *and* $(\mathbf{O}^n)$ *be two sequences of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that H-converge to the homogenised tensors* $\mathbf{M}$ *and* $\mathbf{O}$*, respectively. Furthermore, assume that, for any* n*,*

$$(\forall\, \boldsymbol{\xi} \in \mathrm{Sym}) \qquad \mathbf{M}^n \boldsymbol{\xi} : \boldsymbol{\xi} \leqslant \mathbf{O}^n \boldsymbol{\xi} : \boldsymbol{\xi}\ .$$

Then the homogenised limits are also ordered:

$$(\forall\, \boldsymbol{\xi} \in \mathrm{Sym}) \qquad \mathbf{M} \boldsymbol{\xi} : \boldsymbol{\xi} \leqslant \mathbf{O} \boldsymbol{\xi} : \boldsymbol{\xi}\ .$$

∎

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *that either converges strongly to a limit tensor* $\mathbf{M}$ *in* $\mathrm{L}^1(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, or converges to* $\mathbf{M}$ *almost everywhere in* Ω*. Then,* $\mathbf{M}^n$ *also H-converges to* $\mathbf{M}$*.* ∎

Theorem. *Let* $F = \{f_n : n \in \mathbf{N}\}$ *be a dense countable family in* $\mathrm{H}^{-2}(\Omega)$, **M** *and* **O** *tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$*, and* (u_n)*,* (v_n) *sequences of solutions to*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}\nabla\nabla u_n) = f_n \\ u_n \in \mathrm{H}^2_0(\Omega) \end{cases}$$

and

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{O}\nabla\nabla v_n) = f_n \\ v_n \in \mathrm{H}^2_0(\Omega) \end{cases} .$$

Then,

$$d(\mathbf{M}, \mathbf{O}) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|u_n - v_n\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{M}\nabla\nabla u_n - \mathbf{O}\nabla\nabla v_n\|_{\mathrm{H}^{-1}(\Omega;\mathrm{Sym})}}{\|f_n\|_{\mathrm{H}^{-2}(\Omega)}}$$

is a metric on $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *and H-convergence is equivalent to the convergence with respect to* d*.*

∎

Correctors

Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a limit $\mathbf{M}$, and $(w_n^{ij})_{1\le i,j\le d}$ a family of test functions satisfying

$$\begin{aligned} w_n^{ij} &\rightharpoonup \frac{1}{2} x_i x_j && \text{in} \quad \mathrm{H}^2(\Omega) \\ \mathbf{M}^n \nabla\nabla w_n^{ij} &\rightharpoonup \cdots && \text{in} \quad \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathsf{Sym}) \\ \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla w_n^{ij}) &\to \cdots && \text{in} \quad \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega). \end{aligned}$$

The sequence of tensors $\mathbf{W}^n$ defined by $\mathbf{W}^n_{ijkm} = [\nabla\nabla w_n^{km}]_{ij}$ is called the sequence of correctors.

It is unique, indeed:

Theorem. *Let $(\mathbf{M}^n)$ be a sequence of tensors in $\mathfrak{M}_2(\alpha, \beta; \Omega)$ that H-converges to a tensor $\mathbf{M}$. A sequence of correctors $(\mathbf{W}^n)$ is unique in the sense that, if there exist two sequences of correctors $(\mathbf{W}^n)$ and $(\tilde{\mathbf{W}}^n)$, their difference $(\mathbf{W}^n - \tilde{\mathbf{W}}^n)$ converges strongly to zero in $\mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$.* ∎

Corrector result

Theorem. *Let* $(\mathbf{M}^n)$ *be a sequence of tensors in* $\mathfrak{M}_2(\alpha, \beta; \Omega)$ *which H-converges to* $\mathbf{M}$*. For* $f \in \mathrm{H}^{-2}_{\mathrm{loc}}(\Omega)$*, let* (u_n) *be the solution of*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M}^n \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

and let u *be the weak limit of* (u_n) *in* $\mathrm{H}^2_0(\Omega)$*, i.e. the solution of the homogenised equation*

$$\begin{cases} \operatorname{div}\operatorname{div}(\mathbf{M} \nabla\nabla u) = f \quad \text{in} \quad \Omega \\ u \in \mathrm{H}^2_0(\Omega)\,. \end{cases}$$

Then $\mathbf{R}_n := \nabla\nabla u_n - \mathbf{W}^n \nabla\nabla u \to \mathbf{0}$ *strongly in* $\mathrm{L}^1_{\mathrm{loc}}(\Omega; \mathrm{Sym})$*.*

∎

Smoothness with respect to a parameter

Theorem. *Let* $\mathbf{M}^n : \Omega \times P \to \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})$ *be a sequence of tensors, such that* $\mathbf{M}^n(\cdot, p) \in \mathfrak{M}_2(\alpha, \beta; \Omega)$*, for* $p \in P$*. Assume that* $p \mapsto \mathbf{M}^n(\cdot, p)$ *is of class* C^k *from* P *to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*, with derivatives (up to order* k*) being equicontinuous on every compact set* $K \subseteq P$*:*

$$(\forall K \in \mathcal{K}(P))\,(\forall \varepsilon > 0)(\exists \delta > 0)(\forall p, q \in K)(\forall n \in \mathbf{N})(\forall i \leq k)$$
$$|p - q| < \delta \Rightarrow \|(\mathbf{M}^n)^{(i)}(\cdot, p) - (\mathbf{M}^n)^{(i)}(\cdot, q)\|_{\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))} < \varepsilon.$$

Then there is a subsequence $(\mathbf{M}^{n_k})$ *such that for every* $p \in P$

$$\mathbf{M}^{n_k}(\cdot, p) \xrightarrow{H} \mathbf{M}(\cdot, p) \quad \text{in} \quad \mathfrak{M}_2(\alpha, \beta; \Omega)$$

and $p \mapsto \mathbf{M}(\cdot, p)$ *is a* C^k *mapping from P to* $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$*.* ∎

In particular, the above is valid for $k = \infty$ and $k = \omega$ (the analytic functions).

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Consider a sequence of problems

$$\begin{cases} \mathsf{div}\,\mathsf{div}\,(\mathbf{M}^n_\gamma \nabla\nabla u_n) = f \quad \text{in} \quad \Omega \\ u_n \in \mathrm{H}^2_0(\Omega)\,, \end{cases}$$

where we assume that the coefficients are a small perturbation of a given continuous tensor function $\mathbf{A}_0$, for small γ

$$\mathbf{M}^n_\gamma := \mathbf{A}_0 + \gamma \mathbf{B}^n + \gamma^2 \mathbf{C}^n + o(\gamma^2)\ ,$$

where $\mathbf{B}^n, \mathbf{C}^n \overset{*}{\longrightarrow} \mathbf{O}$ in $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym}))$. For small γ we, in fact, we can assume that the function is analytic in γ.
Then (after passing to a subsequence if needed)

$$\mathbf{M}^n_\gamma \xrightarrow{H} \mathbf{M}^\infty_\gamma = \mathbf{A}_0 + \gamma \mathbf{B}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2)\ ;$$

the limit being measurable in $\mathbf{x}$, and analytic in γ.

Periodic case

- Let Y be the d-dimensional torus, $\mathbf{M} \in \mathrm{L}^\infty(Y; \mathcal{L}(\mathrm{Sym}, \mathrm{Sym})) \cap \mathfrak{M}_2(\alpha, \beta; Y)$
- Assume $\mathbf{M}^n(\mathbf{x}) := \mathbf{M}(n\mathbf{x}), \mathbf{x} \in \Omega \subseteq \mathbf{R}^d$ (projection of $\mathbf{R}^d$ to Y assumed)
- $\mathrm{H}^2(Y)$ consists of 1-periodic functions, with the norm taken over the fundamental period
- $\mathrm{H}^2(Y)/\mathbf{R}$ is equipped with the norm $\|\nabla\nabla \cdot\|_{\mathrm{L}^2(Y)}$
- $\mathbf{E}_{ij}, 1 \leqslant i, j \leqslant d$ are $\mathrm{M}_{d\times d}$ matrices defined as

$$[\mathbf{E}_{ij}]_{kl} = \begin{cases} 1, & \text{if } i = j = k = l \\ \frac{1}{2}, & \text{if } i \neq j, (k,l) \in \{(i,j),(j,i)\} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem. *$(\mathbf{M}^n)$ H-converges to a constant tensor $\mathbf{M}^\infty \in \mathfrak{M}_2(\alpha, \beta; \Omega)$ defined as*

$$m^\infty_{klij} = \int_Y \mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x})) : (\mathbf{E}_{kl} + \nabla\nabla w_{kl}(\mathbf{x}))\, d\mathbf{x},$$

where (w_{ij}) is the family of unique solutions in $\mathrm{H}^2(Y)/\mathbf{R}$ of

$$\begin{cases} \mathrm{div}\,\mathrm{div}\,(\mathbf{M}(\mathbf{x})(\mathbf{E}_{ij} + \nabla\nabla w_{ij}(\mathbf{x}))) = 0 \text{ in Y} \\ \mathbf{x} \to w_{ij}(\mathbf{x}) \quad \textit{is } Y\textit{-periodic.} \end{cases}$$

∎

Small-amplitude assumptions

Theorem. *Let* $\mathbf{A}_0 \in \mathcal{L}(\mathrm{Sym};\mathrm{Sym})$ *be a constant coercive tensor,* $\mathbf{B}^n(\mathbf{x}) := \mathbf{B}(n\mathbf{x})$, $\mathbf{x} \in \Omega$, *where* $\Omega \subseteq \mathbf{R}^d$ *is a bounded, open set, and* $\mathbf{B}$ *is a Y-periodic,* L^∞ *tensor function, satisfying* $\int_Y \mathbf{B}(\mathbf{x})\, d\mathbf{x} = 0$. *Then*

$$\mathbf{M}_\gamma^n(\mathbf{x}) := \mathbf{A}_0 + \gamma \mathbf{B}^n(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

H-converges (for any small γ*) to a tensor* $\mathbf{M}_\gamma := \mathbf{A}_0 + \gamma^2 \mathbf{C}_0 + o(\gamma^2)$ *, where*

$$\begin{aligned}\mathbf{C}_0 \mathbf{E}_{mn} : \mathbf{E}_{rs} = (2\pi i)^2 \sum_{\mathbf{k}\in J} a_{-\mathbf{k}}^{mn} \mathbf{B}_\mathbf{k} (\mathbf{k} \otimes \mathbf{k}) : \mathbf{E}_{rs} + \\ + (2\pi i)^4 \sum_{\mathbf{k}\in J} a_{\mathbf{k}}^{mn} a_{-\mathbf{k}}^{rs} \mathbf{A}_0 (\mathbf{k} \otimes \mathbf{k}) : \mathbf{k} \otimes \mathbf{k} + \\ + (2\pi i)^2 \sum_{\mathbf{k}\in J} a_{-\mathbf{k}}^{rs} \mathbf{B}_\mathbf{k} \mathbf{E}_{mn} : \mathbf{k} \otimes \mathbf{k}\,,\end{aligned}$$

with $m, n, r, s \in \{1, 2, \cdots, d\}$, $J := \mathbf{Z}^d \setminus \{0\}$, *and*

$$a_\mathbf{k}^{mn} = -\frac{\mathbf{B}_\mathbf{k} \mathbf{E}_{mn} \mathbf{k} \cdot \mathbf{k}}{(2\pi i)^2 \mathbf{A}_0 (\mathbf{k} \otimes \mathbf{k}) : (\mathbf{k} \otimes \mathbf{k})}, \quad \mathbf{k} \in J,$$

and $\mathbf{B}_\mathbf{k}$ *are the Fourier coefficients of function* $\mathbf{B}$.

∎

Conjecture in the general case

Theorem. *The effective conductivity matrix* $\mathbf{M}_\gamma^\infty$ *admits the expansion*

$$\mathbf{M}_\gamma^\infty(\mathbf{x}) = \mathbf{A}_0(\mathbf{x}) + \gamma^2 \mathbf{C}_0(\mathbf{x}) + o(\gamma^2) \ ,$$

where the quadratic correction $\mathbf{C}_0$ *can be computed from the H-measure associated to a subsequence of* $\mathbf{B}^n$*.* ∎

Thank you for your attention.