Bounded operators on mixed-norm Lebesgue spaces

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Pseudodifferential operators

Mixed-norm Lebesgue spaces

Hörmander-Mihlin theorem

Continuity of linear operators on mixed-norm Lebesgue spaces

Symbols and operators

$S^m_{\rho,\delta,N,N'}$ … for $|\alpha| \le N, |\beta| \le N'$ it holds

$$(\forall x \in \mathbf{R}^d)(\forall \xi \in \mathbf{R}^d) \quad |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|} ,$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$

norm: $|\sigma|^{(m,\rho,\delta)}_{N,N'} = \max\limits_{|\alpha|\le N, |\beta|\le N'} \sup\limits_{x,\xi\in\mathbf{R}^d} \frac{|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)|}{\langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}}$

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For $\sigma \in S^m_{\rho,\delta,N,N'}$ we denote the corresponding pseudodifferential operator T_σ by

$$T_\sigma \varphi(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x,\xi) \hat{\varphi}(\xi) \; đ\xi, \; \varphi \in \mathcal{S}(\mathbf{R}^d),$$

where $đ\xi = (2\pi)^{-d} d\xi$.

Known continuity results

Our starting point is a famous result by Coifman and Meyer: for $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$ and $m = 0$ it is enough to have $N, N' > \frac{d}{2}$ to obtain the continuity on $L^2(\mathbf{R}^d)$.

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Also, in the smooth case we have the following necessary and sufficient condition for the continuity on $L^p(\mathbf{R}^d)$ spaces:

$$m \leq -d(1-\rho)\left|\frac{1}{2} - \frac{1}{p}\right|.$$

Mixed-norm Lebesgue spaces

[BENEDEK, PANZONE (1961)]

$L^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in [1, \infty\rangle^d$ is a space of measurable complex functions f on $\mathbf{R}^d$,

$$\|f\|_{\mathbf{p}} = \left(\int \cdots \left(\int \left(\int |f(x_1, \ldots, x_d)|^{p_1} \, dx_1 \right)^{\frac{p_2}{p_1}} dx_2 \right)^{\frac{p_3}{p_2}} \cdots dx_d \right)^{\frac{1}{p_d}} < \infty.$$

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$$\mathbf{p}' = (p_1', \ldots, p_d'), \quad \frac{1}{p_i} + \frac{1}{p_i'} = 1$$

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Some facts:

(a) $\mathcal{S} \hookrightarrow L^{\mathbf{p}}(\mathbf{R}^d)$,
(b) $\mathcal{S}$ is dense in $L^{\mathbf{p}}(\mathbf{R}^d)$, for $\mathbf{p} \in [1,\infty\rangle^d$,
(c) $L^{\mathbf{p}'}(\mathbf{R}^d)$ is topological dual of $L^{\mathbf{p}}(\mathbf{R}^d)$, for $\mathbf{p} \in [1,\infty\rangle^d$,
(d) $L^{\mathbf{p}}(\mathbf{R}^d) \hookrightarrow \mathcal{S}'$.

Basic results

We use some generalizations of classical results:

Theorem 1. (dominated convergence for $L^{\mathbf{p}}(\mathbf{R}^d)$ spaces, $\mathbf{p} \in [1, \infty\rangle^d$) *Let (f_n) be a sequence of measurable functions. If $f_n \longrightarrow f$ (ae), and if there exists $G \in L^{\mathbf{p}}(\mathbf{R}^d)$ such that $|f_n| \leqslant G$ (ae), $n \in \mathbf{N}$, then $\|f_n - f\|_{\mathbf{p}} \longrightarrow 0$.* ∎

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Theorem 2. **(Minkowski inequality for integrals)** *For* $\mathbf{p} \in [1, \infty]^{d_1}$ *and* $f \in L^{(\mathbf{p},1,\ldots,1)}(\mathbf{R}^{d_1+d_2})$ *we have*

$$\Big\| \int_{\mathbf{R}^{d_2}} f(\mathbf{x}, \mathbf{y})\, d\mathbf{y} \Big\|_{\mathbf{p}} \leqslant \int_{\mathbf{R}^{d_2}} \|f(\cdot, \mathbf{y})\|_{\mathbf{p}}\, d\mathbf{y}.$$

∎

Basic results (cont.)

Theorem 3. (Hölder inequality) *For* $\mathbf{p} \in [1, \infty]^d$ *we have*

$$\left| \int_{\mathbf{R}^d} f(\mathbf{x}) g(\mathbf{x}) \, d\mathbf{x} \right| \leqslant \|f\|_{\mathbf{p}} \|g\|_{\mathbf{p}'}.$$

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[Benedek, Panzone] proved a converse of Theorem 3:

Theorem 4. *For* $\mathbf{p} \in \langle 1,\infty]^d$ *it follows*

$$\|f\|_{\mathbf{p}} = \sup_{g\in \mathrm{S}_{\mathbf{p}'}} \left|\int f\bar{g}\,d\mathbf{x}\right| = \sup_{g\in \mathrm{S}_{\mathbf{p}'}\cap\mathcal{S}} \left|\int f\bar{g}\,d\mathbf{x}\right|,$$

where $\mathrm{S}_{\mathbf{p}'}$ *is a unit sphere in* $L^{\mathbf{p}'}(\mathbf{R}^d)$.

∎

Fourier multipliers

Theorem 5. *Let* $m \in \mathrm{L}^\infty(\mathbf{R}^d \backslash \{0\})$ *be such that for some* $A > 0$, *and each* $|\boldsymbol{\alpha}| \leqslant [\frac{d}{2}] + 1$ *we have either*
(a) Mihlin condition

$$|\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})| \leqslant A|\boldsymbol{\xi}|^{-|\boldsymbol{\alpha}|} \quad , \text{ or}$$

(b) Hörmander condition

$$\sup_{R>0} R^{-d+2|\boldsymbol{\alpha}|} \int_{R<|\boldsymbol{\xi}|<2R} |\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leqslant A^2 < \infty \, .$$

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Then m *belongs to* $\mathcal{M}_{\mathbf{p}}$*, for each* $\mathbf{p} \in \langle 1, \infty \rangle^d$*, and we have*

$$\begin{aligned} \|m\|_{\mathcal{M}_{\mathbf{p}}} &\leqslant \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max(p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}})(A + \|m\|_{\mathrm{L}^\infty}) \\ &\leqslant c' \prod_{j=0}^{d-1} \max(p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}})(A + \|m\|_{\mathrm{L}^\infty}) \, , \end{aligned}$$

where c *and* c' *depend only on* d*.*

∎

Generalization of the Marcinkiewicz interpolation theorem

Take $l \in \{0, \dots, (d-1)\}$ and split $x = (\bar{x}, x') = (x_1, \dots, x_l; x_{l+1}, \dots, x_d)$.
Next define $\|f\|_{\bar{\mathbf{p}},\, p} = \|f\|_{(\bar{\mathbf{p}},\, p, \dots,\, p)}$ and also a distribution function:

$$\lambda_f(\alpha) = \lambda(f; \alpha) = \text{vol}\{\mathbf{x} \in \mathbf{R}^d : |f(\mathbf{x})| > \alpha\}.$$

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Lemma 1. *Assume that for a linear operator $T : L_c^\infty(\mathbf{R}^d) \to L_{loc}^1(\mathbf{R}^d)$, and some $\bar{\mathbf{p}} \in \langle 1, \infty \rangle^m$ and $q \in \langle 1, \infty \rangle$ there exist $c_1, c_q > 0$ such that for an arbitrary $\alpha > 0$ and $f \in \mathrm{L}_c^\infty(\mathbf{R}^d)$ we have:*

$$\lambda(\|Tf\|_{\bar{\mathbf{p}}}; \alpha) \leqslant c_1 \alpha^{-1} \|f\|_{\bar{\mathbf{p}}, 1} ,$$
$$\|Tf\|_{\bar{\mathbf{p}}, q} \leqslant c_q \|f\|_{\bar{\mathbf{p}}, q} .$$

Then for an arbitrary $p \in \langle 1, q \rangle$ and $f \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ it follows

$$\|Tf\|_{\bar{\mathbf{p}}, p} \leqslant 8(p-1)^{-\frac{1}{p}} (c_1 + c_q) \|f\|_{\bar{\mathbf{p}}, p} .$$

∎

The Caldéron-Zygmund decomposition

The first assumption of the previous lemma could be omitted (under the assumptions of the Hörmander-Mihlin theorem) using the next lemma, where a *dyadic cube* in $\mathbf{R}^d$ is a product of semi-open intervals, i.e. the set of the form (for $k, m_1, \ldots, m_d \in \mathbf{Z}$)

$$[2^k m_1, 2^k (m_1 + 1)\rangle \times \cdots \times [2^k m_d, 2^k (m_d + 1)\rangle \ .$$

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Lemma 2. *Let* $f \in \mathrm{L}^1(\mathbf{R}^d)$ *and* $\alpha > 0$. *Then there exist functions* g *and* b *on* $\mathbf{R}^d$ *satisfying:*

a) $f = g + b$,

b) $\|g\|_{\mathrm{L}^1} \leqslant \|f\|_{\mathrm{L}^1}$ *and* $\|g\|_{\mathrm{L}^\infty} \leqslant 2^d\alpha$,

c) $b = \sum_{k=1}^\infty b_k$, *where each* b_k *is supported in a dyadic cube* Q_k $(Q_j \cap Q_k = \emptyset$ *for* $j \neq k)$,

d) $\int_{Q_k} b_k(\mathbf{x})\, d\mathbf{x} = 0$,

e) $\|b_k\|_{\mathrm{L}^1} \leqslant 2^{d+1}\alpha\,\mathrm{vol}Q_k$, *and*

f) $\sum_{k=1}^\infty \mathrm{vol}Q_k \leqslant \alpha^{-1}\|f\|_{\mathrm{L}^1}$.

∎

The first ΨDO result

Theorem 6. *Let* $\sigma \in S^0_{1,0}$*, with compact support in* x*. Then* T_σ *is bounded on* $L^{\mathbf{p}}(\mathbf{R}^d)$*,* $\mathbf{p} \in \langle 1, \infty \rangle^d$*.*

∎

The general framework

We define (for each $t > 0$ and $y' \in \mathbf{R}^{d-l}$):

$$\mathcal{F}_{l,t}^{y'} := \left\{ f \in L^1_{loc}(\mathbf{R}^d) : \text{supp } f \subseteq \mathbf{R}^l \times \{x' : |x' - y'|_\infty \leq t\} \ \ \& \ \int_{\mathbf{R}^{d-l}} f(\bar{x}, x')\, dx' = 0 \ (\text{ae } \bar{x} \in \mathbf{R}^l) \right\}.$$

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Theorem 7. *Assume that $A, A^* : L_c^\infty(\mathbf{R}^d) \to L^1_{loc}(\mathbf{R}^d)$ are formally adjoint linear operators, i.e. such that*

$$(\forall\, \varphi, \psi \in C_c^\infty(\mathbf{R}^d)) \quad \int_{\mathbf{R}^d} (A\varphi)\overline{\psi} = \int_{\mathbf{R}^d} \varphi \overline{A^*\psi}.$$

Furthermore, let us assume that (both for $T = A$ and $T = A^*$) there exist constants $N > 1$ and $c_1 > 0$ satisfying

$$(\forall\, l \in \{0, \dots, (d-1)\})(\forall\, x_0' \in \mathbf{R}^{d-l})(\forall\, t > 0)$$

$$\int_{|x' - x_0'|_\infty > Nt} \|Tf(\cdot, x')\|_{\bar{\mathbf{p}}}\, dx' \leq c_1 \|f\|_{\bar{\mathbf{p}},1} \;,$$

for any function $f \in L_c^\infty(\mathbf{R}^d) \cap \mathcal{F}_{l,t}^{x_0'}$ and any $\bar{\mathbf{p}} \in \langle 1, \infty\rangle^l$. ∎

The general framework - cont.

Theorem 7. *If for some $q \in \langle 1, \infty \rangle$ an operator A has a continuous extension to an operator from $L^q(\mathbf{R}^d)$ to itself with the norm c_q, then A can be extended by the continuity to an operator from $L^{\mathbf{p}}(\mathbf{R}^d)$ to itself for any $\mathbf{p} \in \langle 1, \infty \rangle^d$, with the norm*

$$\begin{aligned} \|A\|_{L^{\mathbf{p}} \to L^{\mathbf{p}}} &\leq \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max(p_{d-j}, (p_{d-j} - 1)^{-1/p_{d-j}})(c_1 + c_q) \\ &\leq c' \prod_{j=0}^{d-1} \max(p_{d-j}, (p_{d-j} - 1)^{-1/p_{d-j}})(c_1 + c_q), \end{aligned}$$

where c and c' are constants depending only on N and d. ∎

The second ΨDO result

Theorem 8. *Let* $\sigma \in S^0_{1,\delta}$, $\delta \in [0,1\rangle$. *Then* T_σ *is bounded on* $L^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in \langle 1, \infty\rangle^d$.

■

Integral operators

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$$(\exists C > 0) \int_{\mathbf{R}^d} |K(\mathbf{x},\mathbf{y})|\, d\mathbf{x} < C \text{ (ae } \mathbf{y}), \quad \int_{\mathbf{R}^d} |K(\mathbf{x},\mathbf{y})|\, d\mathbf{y} < C \text{ (ae } \mathbf{x}).$$

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Sufficient conditions for continuity on $L^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in \langle 1, \infty \rangle^d$:

$$\int_{\mathbf{R}^d} \|K(\cdot, \cdot - \mathbf{z})\|_{L^\infty}\, d\mathbf{z} < \infty, \quad \int_{\mathbf{R}^d} \|K(\cdot - \mathbf{z}, \cdot)\|_{L^\infty}\, d\mathbf{z} < \infty.$$

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The connection between those conditions = ?

Properties of the kernel

We have $\sigma(x,\cdot) \in \mathcal{S}'(\mathbf{R}^d)$ and so there is a $k(x,\cdot) \in \mathcal{S}'(\mathbf{R}^d)$ such that $\widehat{k(x,\cdot)} = \sigma(x,\cdot)$. Then we can write

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$$T_\sigma \varphi(x) = k(x,\cdot) * \varphi \,.$$

Lemma 3. *Let $\sigma \in S^m_{\rho,\delta,N,N'}$, $\rho > 0$. Then the kernel $k(x,z)$ satisfies*

$$|\partial_x^\alpha \partial_z^\beta k(x,z)| \le C_{\alpha,\beta,L} \cdot |z|^{-d-m-\delta|\alpha|-|\beta|-L}, \quad z \ne 0 \,,$$

for all $|\alpha| \le N$, $|\beta| \ge 0$ and

$$L \ge (1-\rho)\Big(\Big\lfloor \frac{d+m+\delta|\alpha|+|\beta|}{\rho} \Big\rfloor + 1\Big)^+$$

such that $N' \ge d+m+\delta|\alpha|+|\beta|+L > 0$ and $N' > \frac{d+m+\delta|\alpha|+|\beta|}{\rho}$, and where $C_{\alpha,\beta,L}$ is a constant depending only on α, β and L. ∎

The newest ΨDO result

Theorem 9. *Let* $\sigma \in S^m_{\rho,\delta,N,N'}$, $[0,1\rangle \ni \delta \le \rho \in \langle 0,1]$ *and*

$$m \le -(1-\rho)(d+1+\rho)\,.$$

If

$$N > \frac{(3-\delta)d + (5-\delta)(1-\delta)}{(1-\delta)^2}, \quad N' > 6d + 12\,,$$

then T_σ *is bounded on* $L^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in \langle 1,\infty\rangle^d$. ∎