Basic calculus of pseudodifferential operators with nonsmooth symbols

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Symbol classes

$S^m_{\rho,\delta,N,N'}$ ... for $|\alpha| \leq N, |\beta| \leq N'$ it holds

$$(\forall x \in \mathbf{R}^d)(\forall \xi \in \mathbf{R}^d) \quad |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|},$$

where $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$

norm: $|\sigma|^{(m,\rho,\delta)}_{N,N'} = \max_{|\alpha|\leq N, |\beta|\leq N'} \sup_{x,\xi\in\mathbf{R}^d} \frac{|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)|}{\langle \xi \rangle^{m-\rho|\beta|+\delta|\alpha|}}$

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All in one

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It contains both $S^m_{\rho,\delta,N,N'}$ (as a special case) and $\dot{S}^{q,m}_{\rho,\delta,N,N'}$ (as a subclass).

Notation

For $N, N' \in \mathbf{N}_0$ we define an equivalent family of semi-norms on $\mathcal{S}(\mathbb{R}^d)$ with

$$|\varphi|_{N,N'} = \sup_{|\alpha| \le N, |\beta| \le N'} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta \varphi(x)|,$$

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Together with standard notation for partial derivatives ∂_x^α we also use $D_x^\alpha = (-i)^{|\alpha|}\partial_x^\alpha$ and $\langle D_x \rangle^{2k} = (1 - \triangle_x)^k$, where $\triangle$ is the Laplace operator.

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By $\lfloor x \rfloor$ we denote the largest integer not greater than x, while $\lfloor x \rfloor_2$ is the largest even integer not greater than x.

ΨDO - definition and continuity

For $\sigma \in S^{m}_{\rho,\delta,N,N'}$ or $\sigma \in \dot{S}^{q,m}_{\rho,\delta,N,N'}$ we denote the corresponding pseudodifferential operator T_σ by

$$T_\sigma \varphi(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(\xi)\ đ\xi,\ \ \varphi \in \mathcal{S}(\mathbf{R}^d),$$

where $đ\xi = (2\pi)^{-d} d\xi$.

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Lemma 1. *$\mathcal{F} : \mathcal{S}_{N,N'}(\mathbf{R}^d) \to \mathcal{S}_{N',N-d-1}(\mathbf{R}^d)$ is a linear bounded mapping for $N \geq d+1$. More precisely, there is a constant $C_{N,N'} > 0$ such that*

$$|\hat{\varphi}|_{N',N-d-1} \leq C_{N,N'} |\varphi|_{N,N'} \quad \textit{for all } \varphi \in \mathcal{S}_{N,N'}(\mathbf{R}^d)\,.$$

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Theorem 1. *Let $\sigma \in S^m_{\rho,\delta,N,N'}$. Then T_σ is a bounded mapping from $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}_{N',N}(\mathbf{R}^d)$, and from $\mathcal{S}_{M,M'}(\mathbf{R}^d)$ to $\mathcal{S}_{\min\{N',\, M-d-1\},\min\{N,\, M'-m-d-1\}}(\mathbf{R}^d)$, $M \geq d+1$, $M' \geq m+d+1$. More precisely, there is a constant $C_{k,l} > 0$ such that*

$$|T_\sigma \varphi|_{k,l} \leq C_{k,l} |\sigma|^{(m,\rho,\delta)}_{l,k} |\varphi|_{d+1+k,m+d+1+l}\,,$$

for all $k, l \in \mathbf{N}_0$ for which semi-norms are well-defined.

∎

Remark on $\dot{S}$

Theorem 1 holds also for $\sigma \in \dot{S}^{q,m}_{\rho,\delta,N,N'}$, $q \leq 0$ as in that case we have $\dot{S}^{q,m}_{\rho,\delta,N,N'} \subseteq S^{m}_{\rho,\delta,N,N'}$. For $q > 0$ we cannot estimate $\langle x \rangle^{q}$.

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Goals:
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Goals:

1) to prove that this composition is again a pseudodifferential operator with a symbol in a suitable class and to obtain an exact formula and an asymptotic expansion for its symbol.

2) to do the same for a formal adjoint of T_σ, where $\sigma \in S^{m}_{\rho,\delta,N,N'}$.

Amplitudes

The space of amplitudes $\mathcal{A}^{q,m,\delta}_{N,N'}(\mathbf{R}^d \times \mathbf{R}^d)$, $q, m \in \mathbf{R}$, $\delta \in [0, 1)$, $N, N' \in \mathbf{N}_0$, is the set of functions $a : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{C}$ such that

$$|\partial_y^\alpha \partial_\eta^\beta a(y, \eta)| \leq C_{\alpha,\beta} \langle y \rangle^q \langle \eta \rangle^{m+\delta|\alpha|}$$

uniformly in $y, \eta \in \mathbf{R}^d$ for all $|\alpha| \leq N, |\beta| \leq N'$, and where all partial derivatives are understood to be continuous.

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$\mathcal{A}^{q,m,\delta}_{N,N'}(\mathbf{R}^d \times \mathbf{R}^d)$ is the Banach space with the norm

$$|a|_{\mathcal{A}^{q,m,\delta}_{N,N'}} = \max_{|\alpha| \le N, |\beta| \le N'} \sup_{y,\eta \in \mathbf{R}^d} \frac{|\partial_y^\alpha \partial_\eta^\beta a(y,\eta)|}{\langle y \rangle^q \langle \eta \rangle^{m+\delta|\alpha|}} .$$

Oscillatory integrals

Theorem 2. *Let* $a \in \mathcal{A}^{q,m,\delta}_{N,N'}(\mathbf{R}^d \times \mathbf{R}^d)$, $q, m \in \mathbf{R}$, $\delta \in [0,1)$, $N, N' \in 2\mathbf{N}_0$, $N > \frac{m+d}{1-\delta}$, $N' > q + d$, *and let* $\chi \in \mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d)$ *with* $\chi(0,0) = 1$. *Then*

$$\iint e^{-iy\eta} a(y,\eta) dy\, đ\eta := \lim_{\epsilon \to 0} \iint \chi(\epsilon y, \epsilon \eta) e^{-iy\eta} a(y,\eta) dy\, đ\eta$$

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exists and

$$\iint e^{-iy\eta} a(y,\eta) dy\, đ\eta = \iint e^{-iy\eta} \langle y \rangle^{-2l'} \langle D_\eta \rangle^{2l'} \left(\langle \eta \rangle^{-2l} \langle D_y \rangle^{2l} a(y,\eta) \right) dy\, đ\eta\,,$$

where $l, l' \in \mathbf{N}_0$ *are chosen so that the integrand is in* $L^1(\mathbf{R}^d \times \mathbf{R}^d)$, *namely* $N \geq 2l > \frac{m+d}{1-\delta}$, $N' \geq 2l' > q + d$. *Moreover, the definition does not depend on* χ *and*

$$\left| \iint e^{-iy\eta} a(y,\eta) dy\, đ\eta \right| \leq C_{q,m,\delta} |a|_{\mathcal{A}^{q,m,\delta}_{2l,2l'}}\,.$$

∎

Oscillatory = iterated

We can notice that the previous theorem simplifies if $q < -d$ or $m < -d$, and that for $q, m < -d$ we actually have an absolutely convergent integral.

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For example, if $q < -d$, we can take $\chi \in \mathcal{S}(\mathbf{R}^d)$ with $\chi(0) = 1$ and equivalently define the oscillatory integral as

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In this case the oscillatory integral is equal to an iterated integral, whenever the latter exists.

For simplicity, in the sequel we sometimes consider only the case $q, m \geq -d$ as the most interesting one.

Change of variables

For $\chi(y,\eta) \in \mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d)$ with $\chi(0,0) = 1$ a function $\chi(A(y,\eta))$, where A is the regular real matrix, has the same properties and so we are allowed to make a linear change of variables $(y,\eta) = A(y',\eta')$ in the oscillatory integral as long as $y\eta = y'\eta'$, in which case we have

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Moreover, this change of variables can be performed without the requirement $y\eta = y'\eta'$ if we replace $y\eta$ in the definition of the oscillatory integral with a general nondegenerate real quadratic form. In that case we are not able to obtain the representation from Theorem 2.

The Fubini theorem

Theorem 3. *Let* $a \in \mathcal{A}^{q,m,\delta}_{N,N'}(\mathbf{R}^{d+k} \times \mathbf{R}^{d+k})$, $q, m \in \mathbf{R}$, $\delta \in [0,1)$ *and* $N, N' \in \mathbf{N}_0$ *with*

$$N \geq \frac{|m| + k + 2}{1 - \delta}, \quad N' \geq |q| + k + 2\,.$$

Then

$$b(y,\eta) := \iint e^{-iy'\eta'} a(y, y', \eta, \eta') dy' đ\eta' \in \mathcal{A}^{q, m+\delta N, 0}_{N-2l, N'-2l'}(\mathbf{R}^d \times \mathbf{R}^d)\,,$$

where integration is with respect to $\mathbf{R}^k \times \mathbf{R}^k$, $2l > |m| + \delta N + k$, $2l' > |q| + k$, *and*

$$\partial_y^\alpha \partial_\eta^\beta b(y,\eta) = \iint e^{-iy'\eta'} \partial_y^\alpha \partial_\eta^\beta a(y, y', \eta, \eta') dy' đ\eta'\,,$$

for $|\alpha| \leq N - 2l$, $|\beta| \leq N' - 2l'$. ∎

The Fubini theorem - cont.

Theorem 3. *Moreover, if* $\delta \in [0, \frac{1}{2})$, $q, m \geq -d$ *and* $N, N' \in 2\mathbf{N}_0$ *with*

$$N > \frac{m + |m| + \max\{d,k\} + d + k + 2}{1 - 2\delta}, \quad N' > q + |q| + \max\{d,k\} + d + k + 2\,,$$

then

$$\iiiint e^{-iy\eta - iy'\eta'} a(y, y', \eta, \eta') dy dy' đ\eta đ\eta'$$
$$= \iint e^{-iy\eta} \Big(\iint e^{-iy'\eta'} a(y, y', \eta, \eta') dy' đ\eta' \Big) dy\, đ\eta\,.$$

∎

Operators with double symbols

$$T_\sigma^D \varphi(x) = \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} \int_{\mathbf{R}^d} e^{i(x-x')\cdot\xi + i(x'-x'')\cdot\xi'} \sigma(x,\xi,x',\xi')\varphi(x'')dx''\,đ\xi'\,dx'\,đ\xi,$$

where $\varphi \in \mathcal{S}(\mathbf{R}^d)$, the integrals have to be understood as iterated integrals and the symbol σ belongs to one of the following two classes.

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$S^{q_1,m_1,q_2,m_2}_{\rho_1,\delta_1,\rho_2,\delta_2,N_1,N_1',N_2,N_2'}$ $\cdots$

$$|\partial_x^\alpha \partial_\xi^\beta \partial_{x'}^{\alpha'} \partial_{\xi'}^{\beta'} \sigma(x,\xi,x',\xi')| \leq C\langle x\rangle^{q_1}\langle\xi\rangle^{m_1-\rho_1|\beta|+\delta_1|\alpha|}\langle x'\rangle^{q_2}\langle\xi'\rangle^{m_2-\rho_2|\beta'|+\delta_2|\alpha'|}$$

In the case $q_1 = q_2 = 0$ we denote this class as $S^{m_1,m_2}_{\rho_1,\delta_1,\rho_2,\delta_2,N_1,N_1',N_2,N_2'}$.

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$\dot{S}^{q_1,m_1,q_2,m_2}_{\rho_1,\delta_1,\rho_2,\delta_2,N_1,N_1',N_2,N_2'}\ \cdots$

$$|\partial_x^\alpha\partial_\xi^\beta\partial_{x'}^{\alpha'}\partial_{\xi'}^{\beta'}\sigma(x,\xi,x',\xi')| \le C\langle x\rangle^{q_1-|\alpha|}\langle\xi\rangle^{m_1-\rho_1|\beta|+\delta_1|\alpha|}\langle x'\rangle^{q_2-|\alpha'|}\langle\xi'\rangle^{m_2-\rho_2|\beta'|+\delta_2|\alpha'|}$$

The Fubini theorem for double symbols

Theorem 4. *Let* $a \in S^{q_1,m_1,q_2,m_2}_{\rho_1,\delta_1,\rho_2,\delta_2,N_1,N_1',N_2,N_2'}$ *with* $N_1, N_2, N_1', N_2' \in 2\mathbf{N}_0$. *If*

$$N_2 > \frac{m_2+d}{1-\delta_2}, \quad N_2' > q_2 + d\,,$$

then

$$b(y,\eta) := \iint e^{-iy'\eta'} a(y,y',\eta,\eta')dy'\,\bar{d}\eta' \in \mathcal{A}^{q_1,m_1,\delta_1}_{N_1,N_1'}(\mathbf{R}^d \times \mathbf{R}^d)\,,$$

and

$$\partial_y^\alpha \partial_\eta^\beta b(y,\eta) = \iint e^{-iy'\eta'} \partial_y^\alpha \partial_\eta^\beta a(y,y',\eta,\eta')dy'\,\bar{d}\eta'\,,$$

for $|\alpha| \le N_1$, $|\beta| \le N_1'$. ∎

Theorem 4. *Moreover, if* $q_1, q_2, m_1, m_2 \geq -d$ *and*

$$N_1, N_2 > \frac{\tilde{m} + (3-\delta)d + 4(1-\delta)}{(1-\delta)^2}, \quad N_1', N_2' > \tilde{q} + 3d + 4\,,$$

where $\tilde{q} = \max\{q_1, q_2, q_1 + q_2\}$, $\tilde{m} = \max\{m_1, m_2, m_1 + m_2\}$ *and* $\delta = \max\{\delta_1, \delta_2\}$, *then*

$$\iiiint e^{-iy\eta - iy'\eta'} a(y, y', \eta, \eta') dy dy' đ\eta đ\eta'$$

$$= \iint e^{-iy\eta} \Big(\iint e^{-iy'\eta'} a(y, y', \eta, \eta') dy' đ\eta' \Big) dy\, đ\eta\,.$$

∎

Asymptotic expansion I

We want to show that for regular enough symbols we have $T_\sigma^D = T_{\sigma_L}$ where

$$\sigma_L(x,\xi) := \iint e^{-iy\eta}\sigma(x,\xi+\eta,x+y,\xi)dy\, đ\eta\,. \qquad (1)$$

In the next two theorems we first derive asymptotic expansions for σ_L.

Asymptotic expansion I

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In the next two theorems we first derive asymptotic expansions for σ_L.

Theorem 5. *Let* $\sigma \in S^{m_1,m_2}_{\rho_1,\delta_1,\rho_2,\delta_2,N_1,N_1',N_2,N_2'}$, $\rho = \min\{\rho_1,\rho_2\}$, $\delta = \max\{\delta_1,\delta_2\}$, *and let* σ_L *be defined by (1). If*

$$N_2 \geq \rho_1 N_1' + |m_1| + (1-\rho_1)(d+2), \qquad N_1' \geq d+2\,,$$

then

$$\sigma_L \in S^{m_1+m_2+\delta_2(|m_1|+d+2)}_{\rho-\rho_1\delta_2,\,\delta+\delta_1\delta_2,\,\left\lfloor \frac{N_2-\rho_1 N_1' - |m_1| - (1-\rho_1)(d+2)}{1+\delta_1}\right\rfloor,\,N_1'-d-2}\,.$$

∎

Theorem 5. *Moreover, if (for some $K \in \mathbf{N}_0$)*

$$\rho_1 > \frac{\delta_2}{1-\delta_2}, \quad N_2 \geq \rho_1 N_1' + |m_1| + (1-\rho_1)(d+2) + K + 1, \quad N_1' \geq K + d + 3\,,$$

then we have

$$\sigma_L(x,\xi) = \sum_{|\gamma| \leq K} \frac{1}{\gamma!} \partial_\xi^\gamma D_{x'}^\gamma \sigma(x,\xi,x,\xi) + r_L^{(K)}(x,\xi)\,,$$

where

$$r_L^{(K)} \in S^{m_1+m_2+\delta_2(|m_1|+d+2)-(\rho_1-\delta_2-\rho_1\delta_2)(K+1)}_{\rho-\rho_1\delta_2,\, \delta+\delta_1\delta_2,\, \left\lfloor \frac{N_2-\rho_1 N_1'-|m_1|-(1-\rho_1)(d+2)-K-1}{1+\delta_1} \right\rfloor,\, N_1'-K-d-3}\,.$$

∎

Asymptotic expansion II

Theorem 6. *Let* $\sigma \in \dot{S}^{q_1,m_1,q_2,m_2}_{\rho_1,\delta_1,\rho_2,\delta_2,N_1,N_1',N_2,N_2'}$, $\rho = \min\{\rho_1,\rho_2\}$, $\delta = \max\{\delta_1,\delta_2\}$, *and let* σ_L *be defined by (1). If*

$$N_2 \geq |m_1| + d + 2, \qquad N_1' \geq |q_2| + \frac{2(N_2 - |m_1|) - (1-\delta_1)(d+2)}{1+\delta_1},$$

then

$$\sigma_L \in \dot{S}^{q_1+q_2,\, m_1+m_2+\delta_2(|m_1|+d+2)}_{0,\,\delta+\delta_1\delta_2,\, \left\lfloor \frac{N_2-|m_1|-d-2}{1+\delta_1} \right\rfloor,\, \left\lfloor N_1'-|q_2|-\frac{2(N_2-|m_1|)-(1-\delta_1)(d+2)}{1+\delta_1} \right\rfloor}.$$

Asymptotic expansion II

Theorem 6. *Let* $\sigma \in \dot{S}^{q_1,m_1,q_2,m_2}_{\rho_1,\delta_1,\rho_2,\delta_2,N_1,N_1',N_2,N_2'}$, $\rho = \min\{\rho_1,\rho_2\}$, $\delta = \max\{\delta_1,\delta_2\}$, *and let* σ_L *be defined by (1). If*

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then

$$\sigma_L \in \dot{S}^{q_1+q_2,\, m_1+m_2+\delta_2(|m_1|+d+2)}_{0,\, \delta+\delta_1\delta_2,\, \left\lfloor \frac{N_2-|m_1|-d-2}{1+\delta_1} \right\rfloor,\, \left\lfloor N_1'-|q_2|-\frac{2(N_2-|m_1|)-(1-\delta_1)(d+2)}{1+\delta_1} \right\rfloor}.$$

Moreover, if (for some $K \in \mathbf{N}_0$*)* $\rho_1 \geq \frac{\delta_2}{1-\delta_2}$ *and*

$$N_2 \geq |m_1|+d+2+(1+\rho_1)(K+1), \quad N_1' \geq |q_2| + \frac{2(N_2-|m_1|)-(1-\delta_1)(d+2)-2(\rho_1-\delta_1)(K+1)}{1+\delta_1},$$

then we have

$$\sigma_L(x,\xi) = \sum_{|\gamma|\leq K} \frac{1}{\gamma!} \partial_\xi^\gamma D_{x'}^\gamma \sigma(x,\xi,x,\xi) + r_L^{(K)}(x,\xi),$$

where

$$r_L^{(K)} \in \dot{S}^{q_1+q_2-K-1,\, m_1+m_2+\delta_2(|m_1|+d+2)-(\rho_1-\delta_2-\rho_1\delta_2)(K+1)}_{0,\, \delta+\delta_1\delta_2,\, \left\lfloor \frac{N_2-|m_1|-d-2-(1+\rho_1)(K+1)}{1+\delta_1} \right\rfloor,\, \left\lfloor N_1'-|q_2|-\frac{2(N_2-|m_1|)-(1-\delta_1)(d+2)-2(\rho_1-\delta_1)(K+1)}{1+\delta_1} \right\rfloor}$$

$T_\sigma^D = T_{\sigma_L}$

Theorem 7. *Let* $\sigma(x,\xi,x',\xi') = \sigma_1(x,\xi)\sigma_2(x',\xi') \in S^{q_1,m_1,q_2,m_2}_{\rho_1,\delta_1,\rho_2,\delta_2,N_1,N_1',N_2,N_2'}$. *If* $N_1', N_2, N_2', M' \in 2\mathbf{N}_0$, $q_1 \le 0$, $q_2 \in [-d,0]$, $m_1, m_2 \ge -d$ *and*

$$N_2 > \frac{m^*+3d+4}{1-\delta_2}, \quad N_2' > 3d+4, \quad N_1' > N_2'+q_2+3d+4, \quad M > 2d+1, \quad M' > \tilde{m}+(1+\delta_2)N_2+3d+4\,,$$

then

$$T_\sigma^D \varphi(x) = T_{\sigma_L}\varphi(x)\,,$$

where $m^* = \max\{|m_1|, |m_1| + m_1 + m_2\}$, $\tilde{m} = \max\{m_1, m_2, m_1 + m_2\}$ *and* $\varphi \in \mathcal{S}_{M,M'}(\mathbf{R}^d)$. ∎

The composition theorem I

Theorem 8. *Let* $\sigma_1 \in S^{m_1}_{\rho_1,\delta_1,N_1,N_1'}$, $\sigma_2 \in S^{m_2}_{\rho_2,\delta_2,N_2,N_2'}$, $m_1, m_2 \geq -d$, $m^* = \max\{|m_1|, |m_1| + m_1 + m_2\}$, $\tilde{m} = \max\{m_1, m_2, m_1 + m_2\}$, $\rho = \min\{\rho_1, \rho_2\}$, $\delta = \max\{\delta_1, \delta_2\}$ *and* $\varphi \in \mathcal{S}_{M,M'}(\mathbf{R}^d)$. *If* $N_1', N_2, N_2', M' \in 2\mathbf{N}_0$ *and*

$$N_2 > \frac{m^*+3d+4}{1-\delta_2}, \quad N_2' > 3d+4, \quad N_1' > N_2'+3d+4, \quad M > 2d+1, \quad M' > \tilde{m}+(1+\delta_2)N_2+3d+4\,,$$

then

$$(T_{\sigma_1} \circ T_{\sigma_2})\varphi(x) = T_{\sigma_1 \# \sigma_2}\varphi(x)\,,$$

where

$$\sigma_1 \# \sigma_2(x, \xi) = \iint e^{-iy\eta} \sigma_1(x, \xi + \eta) \sigma_2(x + y, \xi) dy\, đ\eta\,.$$

The composition theorem I

Theorem 8. *Let* $\sigma_1 \in S^{m_1}_{\rho_1,\delta_1,N_1,N_1'}$, $\sigma_2 \in S^{m_2}_{\rho_2,\delta_2,N_2,N_2'}$, $m_1, m_2 \geq -d$, $m^* = \max\{|m_1|, |m_1| + m_1 + m_2\}$, $\tilde{m} = \max\{m_1, m_2, m_1 + m_2\}$, $\rho = \min\{\rho_1, \rho_2\}$, $\delta = \max\{\delta_1, \delta_2\}$ *and* $\varphi \in \mathcal{S}_{M,M'}(\mathbf{R}^d)$. *If* $N_1', N_2, N_2', M' \in 2\mathbf{N}_0$ *and*

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$$(T_{\sigma_1} \circ T_{\sigma_2})\varphi(x) = T_{\sigma_1\#\sigma_2}\varphi(x)\,,$$

where

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If additionally $N_2 \geq \rho_1 N_1' + |m_1| + (1-\rho_1)(d+2)$, *then*

$$\sigma_1\#\sigma_2 \in S^{m_1+m_2+\delta_2(|m_1|+d+2)}_{\rho-\rho_1\delta_2,\, \delta+\delta_1\delta_2,\, \left\lfloor \frac{N_2-\rho_1 N_1'-|m_1|-(1-\rho_1)(d+2)}{1+\delta_1} \right\rfloor,\, N_1'-d-2}\,.$$

∎

The composition theorem I - cont.

Theorem 8. *Moreover, if (for some $K \in \mathbf{N}_0$)*

$$\rho_1 > \frac{\delta_2}{1-\delta_2}, \quad N_2 \geq \rho_1 N_1' + |m_1| + (1-\rho_1)(d+2) + K + 1, \quad N_1' \geq K + d + 3\,,$$

then we have the following asymptotic expansion:

$$\sigma_1 \# \sigma_2(x,\xi) = \sum_{|\gamma| \leq K} \frac{1}{\gamma!} \partial_\xi^\gamma \sigma_1(x,\xi) D_x^\gamma \sigma_2(x,\xi) + r^{(K)}(x,\xi)\,,$$

where

$$r^{(K)} \in S^{m_1+m_2+\delta_2(|m_1|+d+2)-(\rho_1-\delta_2-\rho_1\delta_2)(K+1)}_{\rho-\rho_1\delta_2,\, \delta+\delta_1\delta_2,\, \left\lfloor \frac{N_2-\rho_1 N_1'-|m_1|-(1-\rho_1)(d+2)-K-1}{1+\delta_1} \right\rfloor,\, N_1'-K-d-3}\,.$$

∎

The composition theorem II

Theorem 9. *Let* $\sigma_1 \in \dot{S}^{q_1,m_1}_{\rho_1,\delta_1,N_1,N_1'}$, $\sigma_2 \in \dot{S}^{q_2,m_2}_{\rho_2,\delta_2,N_2,N_2'}$, $q_1 \le 0$, $q_2 \in [-d,0]$, $m_1, m_2 \ge -d$, $m^* = \max\{|m_1|, |m_1| + m_1 + m_2\}$, $\tilde{m} = \max\{m_1, m_2, m_1 + m_2\}$, $\rho = \min\{\rho_1, \rho_2\}$, $\delta = \max\{\delta_1, \delta_2\}$ *and* $\varphi \in \mathcal{S}_{M,M'}(\mathbf{R}^d)$. *If* $N_1', N_2, N_2', M' \in 2\mathbf{N}_0$ *and*

$$N_2 > \frac{m^*+3d+4}{1-\delta_2}, \quad N_2' > 3d+4, \quad N_1' > N_2' + q_2 + 3d + 4, \quad M > 2d+1, \quad M' > \tilde{m} + (1+\delta_2)N_2 + 3d + 4,$$

then

$$(T_{\sigma_1} \circ T_{\sigma_2})\varphi(x) = T_{\sigma_1 \# \sigma_2}\varphi(x),$$

where

$$\sigma_1 \# \sigma_2(x,\xi) = \iint e^{-iy\eta} \sigma_1(x, \xi + \eta)\sigma_2(x+y, \xi)\, dy\, đ\eta .$$

The composition theorem II

Theorem 9. *Let* $\sigma_1 \in \dot{S}^{q_1,m_1}_{\rho_1,\delta_1,N_1,N_1'}$, $\sigma_2 \in \dot{S}^{q_2,m_2}_{\rho_2,\delta_2,N_2,N_2'}$, $q_1 \leq 0$, $q_2 \in [-d,0]$, $m_1, m_2 \geq -d$, $m^* = \max\{|m_1|, |m_1| + m_1 + m_2\}$, $\tilde{m} = \max\{m_1, m_2, m_1 + m_2\}$, $\rho = \min\{\rho_1, \rho_2\}$, $\delta = \max\{\delta_1, \delta_2\}$ *and* $\varphi \in \mathcal{S}_{M,M'}(\mathbf{R}^d)$. *If* $N_1', N_2, N_2', M' \in 2\mathbf{N}_0$ *and*

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If additionally $N_1' \geq |q_2| + \frac{2(N_2-|m_1|)-(1-\delta_1)(d+2)}{1+\delta_1}$, *then*

$$\sigma_1 \# \sigma_2 \in \dot{S}^{q_1+q_2,\, m_1+m_2+\delta_2(|m_1|+d+2)}_{0,\, \delta+\delta_1\delta_2,\, \left\lfloor \frac{N_2-|m_1|-d-2}{1+\delta_1} \right\rfloor,\, \left\lfloor N_1' - |q_2| - \frac{2(N_2-|m_1|)-(1-\delta_1)(d+2)}{1+\delta_1} \right\rfloor}\,.$$

■

The composition theorem II - cont.

Theorem 9. *Moreover, if (for some $K \in \mathbf{N}_0$) $\rho_1 \geq \frac{\delta_2}{1-\delta_2}$ and*

$$N_2 \geq |m_1|+d+2+(1+\rho_1)(K+1), \quad N_1' \geq |q_2| + \frac{2(N_2-|m_1|)-(1-\delta_1)(d+2)-2(\rho_1-\delta_1)(K+1)}{1+\delta_1},$$

then we have the following asymptotic expansion:

$$\sigma_1 \# \sigma_2(x,\xi) = \sum_{|\gamma|\leq K} \frac{1}{\gamma!} \partial_\xi^\gamma \sigma_1(x,\xi) D_x^\gamma \sigma_2(x,\xi) + r^{(K)}(x,\xi),$$

where

$$r^{(K)} \in \dot{S}^{q_1+q_2-K-1,\, m_1+m_2+\delta_2(|m_1|+d+2)-(\rho_1-\delta_2-\rho_1\delta_2)(K+1)}_{0,\, \delta+\delta_1\delta_2,\, \left\lfloor \frac{N_2-|m_1|-d-2-(1+\rho_1)(K+1)}{1+\delta_1} \right\rfloor,\, \left\lfloor N_1'-|q_2|-\frac{2(N_2-|m_1|)-(1-\delta_1)(d+2)-2(\rho_1-\delta_1)(K+1)}{1+\delta_1} \right\rfloor}$$

∎

The adjoint

Now we define a formal adjoint of the operator with symbol $\sigma \in S^{q,m}_{\rho,\delta,N,N'}$, $q \leq 0$. From Theorem 1 it follows that T_σ maps $\mathcal{S}(\mathbf{R}^d)$ to $\mathcal{S}_{N',N}(\mathbf{R}^d)$. Also, $\mathcal{S}_{N',N}(\mathbf{R}^d) \subseteq L^2(\mathbf{R}^d)$ for $N' > \frac{d}{2}$. This motivates the following definition.

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Definition

Let $\sigma \in S^{q,m}_{\rho,\delta,N,N'}$, $\sigma^* \in S^{q,m'}_{\rho',\delta',M,M'}$, $q \leq 0$, $M', N' > \frac{d}{2}$. Then T_{σ^*} is called a formal adjoint of T_σ if

$$(\forall \varphi_1, \varphi_2 \in \mathcal{S}(\mathbf{R}^d)) \quad \langle T_\sigma \varphi_1 | \varphi_2 \rangle = \langle \varphi_1 | T_{\sigma^*} \varphi_2 \rangle \,, \tag{2}$$

where $\langle \cdot | \cdot \rangle$ is the standard inner product on $L^2(\mathbf{R}^d)$.

The adjoint theorem

Theorem 10. *Let* $\sigma \in S^{q,m}_{\rho,\delta,N,N'}$, $q \in [-d,0]$, $m \geq -d$, $m^* = \max\{|m|, |m|+m\}$. *If* $N, N' \in 2\mathbf{N}_0$, $\delta < \frac{3-\sqrt{5}}{2}$ *and*

$$N > \frac{[m^* + (3-\delta)d + 4(1-\delta)](1-\delta)^2}{1-3\delta+\delta^2}, \quad N' > 2q + 6d + 10\,,$$

then (2) is satisfied for

$$\sigma^*(x,\xi) = \iint e^{-iy\eta}\overline{\sigma(x+y,\xi+\eta)}dy\, đ\eta\,.$$

The adjoint theorem

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If additionally $N \geq \frac{\rho N' + |m| + (1-\rho)(d+2)}{1-\delta}$, *then*

$$\sigma^* \in S^{m+\delta N}_{\rho,\,0,\,\lfloor(1-\delta)N-\rho N'-|m|-(1-\rho)(d+2)\rfloor,\,N'-d-2}\,,$$

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then (2) is satisfied for

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If additionally $N \geq \frac{\rho N' + |m| + (1-\rho)(d+2)}{1-\delta}$, *then*

$$\sigma^* \in S^{m+\delta N}_{\rho,\,0,\,\lfloor(1-\delta)N-\rho N'-|m|-(1-\rho)(d+2)\rfloor,\,N'-d-2}\,,$$

while if additionally $\sigma \in \dot{S}^{q,m}_{\rho,\delta,N,N'}$ *and* $N' \geq |q| + 2((1-\delta)N - |m|) - d - 2$, *then*

$$\sigma^* \in \dot{S}^{q,\,m+\delta N}_{0,\,0,\,\lfloor(1-\delta)N-|m|-d-2\rfloor,\,\lfloor N'-|q|-2((1-\delta)N-|m|)+d+2\rfloor}\,.$$

∎

The adjoint theorem - cont.

Theorem 10. *Moreover, if (for some $K \in \mathbf{N}_0$)*

$$\frac{\rho}{1+\rho} > \frac{\delta}{1-\delta}, \quad N \geq \frac{\rho N' + |m| + (1-\rho)(d+2) + K + 1}{1-\delta}, \quad N' \geq K+d+3\,,$$

then we have the following asymptotic expansion:

$$\sigma^*(x,\xi) = \sum_{|\gamma| \leq K} \frac{1}{\gamma!} \partial_\xi^\gamma D_x^\gamma \overline{\sigma(x,\xi)} + r_*^{(K)}(x,\xi)\,, \tag{3}$$

where

$$r_*^{(K)} \in S^{m+\delta N-\rho(K+1)}_{\rho,\,0,\,\lfloor(1-\delta)N-\rho N'-|m|-(1-\rho)(d+2)-K-1\rfloor,\,N'-K-d-3}\,.$$

The adjoint theorem - cont.

Theorem 10. *Moreover, if (for some $K \in \mathbf{N}_0$)*

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$$r_*^{(K)} \in S^{m+\delta N-\rho(K+1)}_{\rho,\,0,\,\lfloor(1-\delta)N-\rho N'-|m|-(1-\rho)(d+2)-K-1\rfloor,\,N'-K-d-3}\,.$$

In the case $\sigma \in \dot{S}^{q,m}_{\rho,\delta,N,N'}$ the asymptotic expansion (3) is valid also for

$$\tfrac{\rho}{1+\rho} \geq \tfrac{\delta}{1-\delta}, \quad N \geq \tfrac{|m|+d+2+(1+\rho)(K+1)}{1-\delta}, \quad N' \geq |q|+2((1-\delta)N-|m|)-d-2-2\rho(K+1),$$

in which case we obtain

$$r_*^{(K)} \in \dot{S}^{q-K-1,\,m+\delta N-\rho(K+1)}_{0,\,0,\,\lfloor(1-\delta)N-|m|-d-2-(1+\rho)(K+1)\rfloor,\,\lfloor N'-|q|-2((1-\delta)N-|m|)+d+2+2\rho(K+1)\rfloor}\,.$$

∎