Continuity of linear operators on mixed-norm Lebesgue and Sobolev spaces

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MiTPDE

Mixed-norm Lebesgue spaces

[Benedek, Panzone (1961)]

$\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, $\mathbf{p} \in [1, \infty\rangle^d$ is space of measurable complex functions f on $\mathbf{R}^d$,

$$\|f\|_{\mathbf{p}} = \left(\int \cdots \left(\int \left(\int |f(x_1, \ldots, x_d)|^{p_1} \, dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{p_3}{p_2}} \cdots dx_d\right)^{\frac{1}{p_d}} < \infty.$$

If $p_i = \infty$, analogously. $\|\cdot\|_{\mathbf{p}}$ is a norm and $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ is a Banach space.

$$\mathbf{p}' = (p_1', \ldots, p_d'), \quad \frac{1}{p_i} + \frac{1}{p_i'} = 1$$

Some facts:

(a) $\mathcal{S} \hookrightarrow \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$,
(b) $\mathcal{S}$ is dense in $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, for $\mathbf{p} \in [1, \infty\rangle^d$,
(c) $\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)$ is topological dual of $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$, for $\mathbf{p} \in [1, \infty\rangle^d$,
(d) $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \hookrightarrow \mathcal{S}'$.

Basic results

We use some generalizations of classical results:

Theorem 1. (dominated convergence for $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ spaces, $\mathbf{p} \in [1,\infty\rangle^d$) *Let (f_n) be sequence of measurable functions. If $f_n \longrightarrow f$ (ae), and if there exists $G \in \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ such that $|f_n| \leqslant G$ (ae), for $n \in \mathbf{N}$, then $\|f_n - f\|_{\mathbf{p}} \longrightarrow 0$.* ∎

Theorem 2. (Minkowski ineaquality for integrals) *For $\mathbf{p} \in [1,\infty]^{d_1}$ and $f \in \mathrm{L}^{(\mathbf{p},1,\ldots,1)}(\mathbf{R}^{d_1+d_2})$ we have*

$$\Big\| \int_{\mathbf{R}^{d_2}} f(\mathbf{x},\mathbf{y})\, d\mathbf{y} \Big\|_{\mathbf{p}} \leqslant \int_{\mathbf{R}^{d_2}} \|f(\cdot,\mathbf{y})\|_{\mathbf{p}}\, d\mathbf{y}.$$

∎

Basic results (cont.)

Theorem 3. (Hölder ineaqualit y) *For* $\mathbf{p} \in [1,\infty]^d$ *we have*

$$\left|\int_{\mathbf{R}^d} f(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x}\right| \leqslant \|f\|_{\mathbf{p}} \|g\|_{\mathbf{p}'}.$$

∎

[Benedek, Panzone] prove a converse of Theorem 3:

Theorem 4. *For* $\mathbf{p} \in \langle 1,\infty]^d$ *it follows*

$$\|f\|_{\mathbf{p}} = \sup_{g \in \mathrm{S}_{\mathbf{p}'}} \left|\int f \bar{g}\, d\mathbf{x}\right| = \sup_{g \in \mathrm{S}_{\mathbf{p}'} \cap \mathcal{S}} \left|\int f \bar{g}\, d\mathbf{x}\right|,$$

where $\mathrm{S}_{\mathbf{p}'}$ *is a unit sphere in* $\mathrm{L}^{\mathbf{p}'}(\mathbf{R}^d)$.

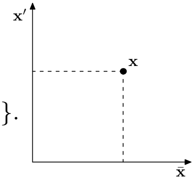
∎

Notation

$$\mathbf{x} = (\bar{\mathbf{x}}, \mathbf{x}'),\ \bar{\mathbf{x}} = (x_1, \dots, x_r),\ \mathbf{x}' = (x_{r+1}, \dots, x_d),\ 0 \leqslant r \leqslant d-1,$$

$$\mathrm{L}^{\bar{\mathbf{p}},\, p}(\mathbf{R}^d) = \mathrm{L}^{(\bar{\mathbf{p}}, p, \dots, p)}(\mathbf{R}^d),\ \|f\|_{\bar{\mathbf{p}},\, p} = \|f\|_{(\bar{\mathbf{p}},\, p, \dots,\, p)},\ \bar{\mathbf{p}} = (p_1, \dots, p_r).$$

If $r = 0$: $\quad \|f(\cdot, \mathbf{x}')\|_{\bar{\mathbf{p}}} = |f(\mathbf{x}')|, \quad \|f\|_{\bar{\mathbf{p}},\, p} = \|f\|_{\mathrm{L}^p}.$



Distribution function:

$$\lambda_f(\alpha) = \lambda(f; \alpha) = \mathsf{vol}\{\mathbf{x} \in \mathbf{R}^d : |f(\mathbf{x})| > \alpha\}.$$

(a) λ_f is non-increasing and right continuous.
(b) If $|f| \leqslant |g|$, then $\lambda_f \leqslant \lambda_g$.
(c) If $|f_n| \nearrow |f|$, then $\lambda_{f_n} \nearrow \lambda_f$.
(d) If $f = g + h$, it follows $\lambda(f; \alpha) \leqslant \lambda(g; \frac{\alpha}{2}) + \lambda(h; \frac{\alpha}{2})$.

Main theorem (hypotheses)

Theorem 5. *Let us assume that linear operators* $A, A^* : \mathrm{L}^\infty_c(\mathbf{R}^d) \to \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$ *satisfy*

$$(\forall\, \varphi, \psi \in \mathrm{C}^\infty_c(\mathbf{R}^d)) \quad \int_{\mathbf{R}^d} (A\varphi)\overline{\psi} = \int_{\mathbf{R}^d} \varphi\overline{A^*\psi}.$$

Furthermore, assume that (for $T = A$ *and* $T = A^*$*) there exist* $N > 1$ *and* $c_1 > 0$ *such that*

$$(\forall\, m \in 0..(d{-}1))(\forall\, \mathbf{x}'_0 \in \mathbf{R}^{d-m})(\forall\, t > 0) \qquad \int\limits_{|\mathbf{x}'-\mathbf{x}'_0|_\infty > Nt} \|Tf(\cdot, \mathbf{x}')\|_{\bar{\mathbf{p}}}\, d\mathbf{x}' \leqslant c_1 \|f\|_{\bar{\mathbf{p}},1},$$

for an arbitrary $f \in \mathrm{L}^\infty_c(\mathbf{R}^d)$ *with properties:*

(a) $\operatorname{supp} f \subseteq \mathbf{R}^m \times \{\mathbf{x}' : |\mathbf{x}' - \mathbf{x}'_0|_\infty \leqslant t\}$,

(b) $(\forall\, \bar{\mathbf{x}} \in \mathbf{R}^m) \quad \int_{\mathbf{R}^{d-m}} f(\bar{\mathbf{x}}, \mathbf{x}')\, d\mathbf{x}' = 0$.

∎

Main theorem (conclusion)

Theorem 5.
Let A has a continuous extension to $\mathrm{L}^q(\mathbf{R}^d)$ with norm c_q for some $q \in \langle 1, \infty\rangle$, then A has a continuous extension also to $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$ for each $\mathbf{p} \in \langle 1, \infty\rangle^d$, with norm

$$\begin{aligned}\|A\|_{\mathrm{L}^{\mathbf{p}} \to \mathrm{L}^{\mathbf{p}}} &\leqslant \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max(p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}})(c_1 + c_q) \\ &\leqslant c' \prod_{j=0}^{d-1} \max(p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}})(c_1 + c_q),\end{aligned}$$

where c and c' depend only on N and d. $\blacksquare$

Main step in the proof

The proof is inductive by using the following lemma.

Lemma 1. *Assume that linear operators* $A, A^* : \mathrm{L}^\infty_c(\mathbf{R}^d) \to \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^d)$ *satisfy assumptions of Theorem 5.*
If A *extends continuously to* $\mathrm{L}^{\bar{\mathbf{p}},\,q}(\mathbf{R}^d)$ *with norm* c_q*, for some* $\bar{\mathbf{p}} \in \langle 1, \infty\rangle^m$ *and* $q \in \langle 1, \infty\rangle$*, then* A *also extends continuously to* $\mathrm{L}^{\bar{\mathbf{p}},\,p}(\mathbf{R}^d)$ *for each* $p \in \langle 1, \infty\rangle$*, with norm*

$$\|A\| \leqslant c \cdot \max(p, (p-1)^{-1/p})(c_1 + c_q),$$

where c *depends only on* N *and* d*.* ∎

Generalization of Marcinkiewicz interpolation theorem

Lemma 2. *Assume that for linear operator* $T : \mathrm{L}_c^\infty(\mathbf{R}^d) \to \mathrm{L}_{\mathrm{loc}}^1(\mathbf{R}^d)$, *and some* $\bar{\mathbf{p}} \in \langle 1, \infty\rangle^m$ *and* $q \in \langle 1, \infty\rangle$ *there exist* $c_1, c_q > 0$ *such that for arbitrary* $\alpha > 0$ *and* $f \in \mathrm{L}_c^\infty(\mathbf{R}^d)$ *we have:*

$$\lambda(\|Tf\|_{\bar{\mathbf{p}}}; \alpha) \leqslant c_1 \alpha^{-1} \|f\|_{\bar{\mathbf{p}},\,1} \,,$$
$$\|Tf\|_{\bar{\mathbf{p}},q} \leqslant c_q \|f\|_{\bar{\mathbf{p}},\,q} \,.$$

Then for arbitrary $p \in \langle 1, q\rangle$ *and* $f \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ *it follows*

$$\|Tf\|_{\bar{\mathbf{p}},\,p} \leqslant 8(p-1)^{-\frac{1}{p}}(c_1 + c_q)\|f\|_{\bar{\mathbf{p}},\,p} \,.$$

∎

Example 1 - Fourier multipliers

Theorem 6. *Let* $m \in \mathrm{L}^\infty(\mathbf{R}^d \backslash \{0\})$ *be such that for some* $A > 0$, *and each* $|\boldsymbol{\alpha}| \leqslant [\frac{d}{2}] + 1$ *we have either*
(a) Mihlin condition

$$|\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})| \leqslant A|\boldsymbol{\xi}|^{-|\boldsymbol{\alpha}|} \quad , \text{ or}$$

(b) Hörmander condition

$$\sup_{R>0} R^{-d+2|\boldsymbol{\alpha}|} \int_{R<|\boldsymbol{\xi}|<2R} |\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}} m(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} \leqslant A^2 < \infty \, .$$

Then m *belongs to* $\mathcal{M}_{\mathbf{p}}$, *for each* $\mathbf{p} \in \langle 1, \infty \rangle^d$, *and we have*

$$\begin{aligned} \|m\|_{\mathcal{M}_{\mathbf{p}}} &\leqslant \sum_{k=1}^{d} c^k \prod_{j=0}^{k-1} \max(p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}})(A + \|m\|_{\mathrm{L}^\infty}) \\ &\leqslant c' \prod_{j=0}^{d-1} \max(p_{d-j}, (p_{d-j}-1)^{-1/p_{d-j}})(A + \|m\|_{\mathrm{L}^\infty}) \, , \end{aligned}$$

where c *and* c' *depends only on* d.

∎

Example 2 - pseudodifferential operators

$a(\mathbf{x},\boldsymbol{\xi}) \in \mathrm{C}^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ is Hörmander symbol of order m ($a \in S^m_{1,\delta}$) if:

$$(\forall\, \mathbf{x} \in \mathbf{R}^d)\ (\forall\, \boldsymbol{\xi} \in \mathbf{R}^d) \quad |\partial_{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}} a(\mathbf{x},\boldsymbol{\xi})| \leqslant C_{\boldsymbol{\alpha},\boldsymbol{\beta}}(1+4\pi^2|\boldsymbol{\xi}|^2)^{\frac{m-|\boldsymbol{\beta}|+\delta|\boldsymbol{\alpha}|}{2}},$$

$\partial_{\boldsymbol{\alpha}}\partial^{\boldsymbol{\beta}} a(\mathbf{x},\boldsymbol{\xi}) := \partial^{\boldsymbol{\alpha}}_{\mathbf{x}}\partial^{\boldsymbol{\beta}}_{\boldsymbol{\xi}} a(\mathbf{x},\boldsymbol{\xi})$, $C_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is constant depending only on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

We definy $a(\cdot, D) : \mathcal{S} \longrightarrow \mathcal{S}$ by

$$(a(\mathbf{x}, D)\varphi)(\mathbf{x}) = \int_{\mathbf{R}^d} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} a(\mathbf{x},\boldsymbol{\xi})\hat{\varphi}(\boldsymbol{\xi})\, d\boldsymbol{\xi}.$$

Adjoint operator $a^*(\cdot, D)$, with symbol

$$a^*(\mathbf{x},\boldsymbol{\xi}) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\eta}}\, \bar{a}(\mathbf{x}-\mathbf{y}, \boldsymbol{\xi}-\boldsymbol{\eta})\, d\mathbf{y}\, d\boldsymbol{\eta},$$

defines an extension $a(\cdot, D) : \mathcal{S}' \longrightarrow \mathcal{S}'$, a pseudodifferential operator of order m, by formula

$$\langle a(\cdot, D)u, \varphi\rangle = \langle u, a^*(\cdot, D)\varphi\rangle.$$

Example 2 - cont.

Theorem 7. *Pseudodifferential operators of class* $\Psi^0_{1,\delta}$*, for an arbitrary* $\delta \in [0,1\rangle$*, are bounded on* $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$*,* $\mathbf{p} \in \langle 1,\infty\rangle^d$*.* ∎

We also get the following corollary and generalisation for operators between mixed-norm Sobolev spaces, defined for $k \in \mathbf{N}_0$ and $\mathbf{p} \in \langle 1,\infty\rangle^d$ by

$$\mathrm{W}^{k,\mathbf{p}}(\mathbf{R}^d) = \Big\{ f \in \mathcal{S}' \ : \ (\forall\, \boldsymbol{\alpha} \in \mathbf{N}_0^d) \quad |\boldsymbol{\alpha}| \leqslant k \quad \Longrightarrow \quad \partial^{\boldsymbol{\alpha}} f \in \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d) \Big\},$$

with the norm

$$\|f\|_{\mathrm{W}^{k,\mathbf{p}}(\mathbf{R}^d)} = \sum_{|\boldsymbol{\alpha}| \leqslant k} \|\partial^{\boldsymbol{\alpha}} f\|_{\mathbf{p}} \,.$$

Corollary. *Let* $\delta \in [0,1\rangle$ *and let* $a(\cdot, D)$ *be a pseudodifferential operator from* $\Psi^m_{1,\delta}$*. Then for any* $\mathbf{p} \in \langle 1,\infty\rangle^d$ *and any integer* $k \geqslant m \in \mathbf{N}_0$ *the operator* $a(\cdot, D) : \mathrm{W}^{k,\mathbf{p}}(\mathbf{R}^d) \longrightarrow \mathrm{W}^{k-m,\mathbf{p}}(\mathbf{R}^d)$ *is bounded.* ∎

Example 3 - integral operators

$$Tf(\mathbf{x}) = \int_{\mathbf{R}^d} K(\mathbf{x},\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}$$

Continuity on $\mathrm{L}^p(\mathbf{R}^d)$ (Schur):

$$(\exists\, C > 0) \int_{\mathbf{R}^d} |K(\mathbf{x},\mathbf{y})|\, d\mathbf{x} < C \text{ (ae } \mathbf{y}), \quad \int_{\mathbf{R}^d} |K(\mathbf{x},\mathbf{y})|\, d\mathbf{y} < C \text{ (ae } \mathbf{x}).$$

Sufficient condition for continuity on $\mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)$:

$$\int_{\mathbf{R}^d} \|K(\cdot, \cdot - \mathbf{z})\|_{\mathrm{L}^\infty}\, d\mathbf{z} < \infty.$$

Connection between those conditions=?

A compactness result – in two steps

By using Theorem 6 (Hörmander – Mihlin) we get

Theorem 8. *Let* $s_0, s_1 \in \mathbf{R}$, $0 < \theta < 1$ *and* $s = (1-\theta)s_0 + \theta s_1$. *Then*

$$\Big(\mathrm{H}^{s_0,\mathbf{p}_0}(\mathbf{R}^d), \mathrm{H}^{s_1,\mathbf{p}_1}(\mathbf{R}^d)\Big)_{[\theta]} = \mathrm{H}^{s,\mathbf{p}}(\mathbf{R}^d)\ ,$$

for any $\mathbf{p}_0, \mathbf{p}_1 \in \langle 1, \infty\rangle^d$, *where* $1/\mathbf{p} = (1-\theta)/\mathbf{p}_0 + \theta/\mathbf{p}_1$. ∎

$$\mathrm{H}^{s,\mathbf{p}}(\mathbf{R}^d) = \Big\{u \in \mathcal{S}' : \mathcal{F}^{-1}(\lambda^s \hat{u}) \in \mathrm{L}^{\mathbf{p}}(\mathbf{R}^d)\Big\}$$

Then we can prove the Rellich-Kondrašov theorem for mixed-norm Sobolev spaces:

Theorem 9. *Let* $\mathbf{p} \in \langle 1, \infty\rangle^d$, $t < s$ *and* $\varphi \in C_c^\infty(\mathbf{R}^d)$. *Assume that* (u_n) *is a bounded sequence in* $\mathrm{H}^{s,\mathbf{p}}(\mathbf{R}^d)$. *Then there exists a subsequence of the given sequence (which we do not relabel) such that* (φu_n) *converges strongly in* $\mathrm{H}^{t,\mathbf{p}}(\mathbf{R}^d)$. ∎