Application of defect distributions to equations with polynomial coefficients

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Motivation - H-measures, H-distributions

- $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$, $n \to \infty$

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Existence of H-measure (Tartar)

There exists a subsequence $(u_{n'})$ and a complex Radon measure μ on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ s. t. for all $\varphi_1(x), \varphi_2(x) \in C_0(\mathbb{R}^d)$, $\psi(\xi) \in C(\mathbb{S}^{d-1})$ we have that

$$\lim_{n' \to \infty} \int_{\mathbb{R}^d} \mathcal{F}(\varphi_1 u_{n'})(\xi) \overline{\mathcal{F}(\varphi_2 u_{n'})}(\xi) \psi\left(\frac{\xi}{|\xi|}\right) d\xi$$

$$= \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \varphi_1(x) \overline{\varphi_2}(x) \psi(\xi) d\mu(x, \xi) = \langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle$$

- $\mathbb{S}^{d-1}$ - unit sphere in $\mathbb{R}^d$

- If $u_n \to 0$ in $L^2(\mathbb{R}^d)$, then $\mu = 0$.
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Localisation principle for H-measures

$$P(x,\xi)\mu(x,\xi) = \sum_{j=1}^{d} A_j(x)\xi_j\, \mu(x,\xi) = 0, \text{i.e. supp}\ \mu \subset chP$$

H-distributions

Theorem (H-measures, equivalent formulation)

Let sequences $u_n, v_n \rightharpoonup 0$ *in* $L^2(\mathbb{R}^d)$. *There exist* $(u_{n'}), (v_{n'})$ *and a complex Radon measure* μ *on* $\mathbb{R}^d \times \mathbb{S}^{d-1}$ *such that for all* $\varphi_1, \varphi_2 \in C_0(\mathbb{R}^d), \psi \in C(\mathbb{S}^{d-1})$

$$\langle \mu, \varphi_1 \overline{\varphi}_2 \psi \rangle := \lim_{n' \to \infty} \langle \varphi_1 u_{n'}, \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})} \rangle.$$

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- H-distributions (Antonić, Mitrović, 2011.) - $L^p - L^q$ spaces, $p = \dfrac{q}{q-1}$, $1 < p < \infty$

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$u_n \rightharpoonup 0$ in $L^p(\mathbb{R}^d)$, $v_n \rightharpoonup 0$ in $L^q(\mathbb{R}^d)$

- H-distributions - $W^{-k,p} - W^{k,q}$, $H^p_{-s} - H^q_{-s}$ spaces, $s \in \mathbb{R}, 1 < p < \infty$ (Aleksić, Pilipović, V.)

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Theorem

If a sequence $u_n \rightharpoonup 0$ weakly in $W^{-k,p}(\mathbb{R}^d)$ and $v_n \rightharpoonup 0$ weakly in $W^{k,q}(\mathbb{R}^d)$, then there exist subsequences $(u_{n'}), (v_{n'})$ and a distribution μ such that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in C^\kappa(\mathbb{S}^{d-1})$, $\kappa = [d/2] + 1$,

$$\lim_{n' \to \infty} \langle \varphi_1 u_{n'} \, , \overline{\mathcal{A}_{\overline{\psi}}(\varphi_2 v_{n'})} \rangle = \langle \mu, \varphi_1 \bar{\varphi}_2 \psi \rangle .$$

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- $\mu \in \mathcal{SE}'(\mathbb{R}^d \times \mathbb{S}^{d-1})$
- $\mathcal{S}(\mathbb{R}^d) \hat{\otimes} \mathcal{E}(\mathbb{S}^{d-1}) = \mathcal{SE}(\mathbb{R}^d \times \mathbb{S}^{d-1})$.

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Let $u_n \rightharpoonup 0$ *in* $W^{-k,p}(\mathbb{R}^d)$*. If for every sequence* $v_n \rightharpoonup 0$ *in* $W^{k,q}(\mathbb{R}^d)$ *the corresponding H-distribution is zero, then for every* $\theta \in \mathcal{S}(\mathbb{R}^d)$*,* $\theta u_n \to 0$ *strongly in* $W^{-k,p}(\mathbb{R}^d)$*,* $n \to \infty$*.*

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Localization property

- $1 < q < d$, $u_n \rightharpoonup 0$ in $W^{-k,p}$, $v_n \rightharpoonup 0$ in $W^{k,q}$
- $\sum_{i=1}^{d} \partial_{x_i}(A_i(x)u_n(x)) = f_n(x)$, $A_i \in \mathcal{S}(\mathbb{R}^d)$, $\theta f_n \to 0$ in $W^{-k-1,p}$, $n \to \infty$ for every $\theta \in \mathcal{S}(\mathbb{R}^d)$

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$$\sum_{j=1}^{d} A_j(x)\xi_j \, \mu(x,\xi) = 0 \text{ in } \mathcal{SE}'(\mathbb{R}^d \times \mathbb{S}^{d-1})$$

$$\text{supp } \mu \subset \text{char } P$$

Weight functions

- Defect distributions - $H_{\Lambda}^{-s,p} - H_{\Lambda}^{s,q}$ spaces, weights $\Lambda = \Lambda(x, \xi)$ (Pilipović, V.)

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- Defect distributions - $H^{-s,p}_{\Lambda} - H^{s,q}_{\Lambda}$ spaces, weights $\Lambda = \Lambda(x, \xi)$ (Pilipović, V.)

Definition

Positive function $\Lambda \in C^\infty(\mathbb{R}^N)$ is a weight function if the following conditions are satisfied:

1. There exist positive constants $1 \leq \mu_0 \leq \mu_1$ and $c_0 < c_1$ such that
$$c_0 \langle z \rangle^{\mu_0} \leq \Lambda(z) \leq c_1 \langle z \rangle^{\mu_1}, \quad z \in \mathbb{R}^N;$$
2. There exists $\omega \geq \mu_1$ such that for any $\alpha \in \mathbb{N}_0^N$ and $\gamma \in \mathbb{K}_N \equiv \{0, 1\}^N$
$$|z^\gamma \partial^{\alpha+\gamma} \Lambda(z)| \leq C_{\alpha,\gamma} \Lambda(z)^{1 - \frac{1}{\omega}|\alpha|}, \quad z \in \mathbb{R}^N.$$

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2. Multi-quasi-elliptic polynomial:

$$\Lambda_{\mathcal{P}}(z) = \Big(\sum_{\alpha \in V(\mathcal{P})} z^{2\alpha} \Big)^{\frac{1}{2}}, \; z \in \mathbb{R}^N.$$

Here $\mathcal{P}$ is a given complete polyhedron with the set of vertices $V(\mathcal{P})$.

Definition

Let $m \in \mathbb{R},\ \rho \in (0, 1/\omega]$. We denote by $M\Gamma^m_{\rho,\Lambda}$ the space of functions $a \in C^\infty(\mathbb{R}^{2d})$ such that for all $\alpha, \beta \in \mathbb{N}_0^d$, $\gamma_1, \gamma_2 \in \{0, 1\}^d$ it holds that

$$|x^{\gamma_1}\xi^{\gamma_2}\partial_\xi^{\alpha+\gamma_2}\partial_x^{\beta+\gamma_1}a(x,\xi)| \le C\Lambda(x,\xi)^{m-\rho|\alpha+\beta|},\ \ (x,\xi) \in \mathbb{R}^{2d}.$$

We equip $M\Gamma^m_{\rho,\Lambda}$ with the family of norms

$$\|a\|_{M\Gamma^m_k} = \sup_{|\alpha|+|\beta|\leq k, \gamma\in\mathbb{K}} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{|x^{\gamma_1}\xi^{\gamma_2}\partial_\xi^{\alpha+\gamma_2}\partial_x^{\beta+\gamma_1} a(x,\xi)|}{\Lambda(x,\xi)^{m-\rho|\alpha+\beta|}},$$

where $k \in \mathbb{N}_0$, $\gamma = (\gamma_1, \gamma_2)$, $\gamma_i \in \mathbb{K}_d, \alpha, \beta \in \mathbb{N}_0^d$.
Pseudo-differential operator T_a with a symbol $a \in M\Gamma^m_{\rho,\Lambda}$ is defined by

$$T_a u(x) := \int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\hat{u}(\xi) đ\xi, \quad u \in \mathcal{S}(\mathbb{R}^d).$$

Let $\Lambda(x,\xi)$ be a weight function, $s \in \mathbb{R}$, $1 < p < \infty$. We denote by $H^{s,p}_{\Lambda}(\mathbb{R}^d)$ the space of all $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $T_{\Lambda^s} u \in L^p(\mathbb{R}^d)$.
Since $\Lambda(x,\xi)^s$ is elliptic of order s there exists an operator $T_b \in ML^{-s}_{\rho,\Lambda}$ such that

$$T_b T_{\Lambda^s} = I + R_s,$$

where R_s is a regularizing operator. We define norm on $H^{s,p}_{\Lambda}$ in the following manner:

$$\|u\|_{s,p,\Lambda} = \|T_{\Lambda^s} u\|_{L^p} + \|R_s u\|_{L^p}.$$

With this norm $H^{s,p}_{\Lambda}(\mathbb{R}^d)$ becomes a Banach space.

Theorem

If $b \in M\Gamma^m_{1/\omega,\Lambda}$, then $T_b : H^{s+m,p}_\Lambda(\mathbb{R}^d) \to H^{s,p}_\Lambda(\mathbb{R}^d)$ continuously for $s, m \in \mathbb{R}$ and $1 < p < \infty$. We have the following estimate

$$\|T_b u\|_{H^{s,p}_\Lambda} \leq C\|b\|_{M\Gamma^m_k}\|u\|_{H^{s+m,p}_\Lambda},$$

for some $k \in \mathbb{N}, k > 2d$.

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Theorem (Lizorkin-Marcinkiewicz)

Let $m(\xi)$ be continuous together with derivatives $\partial^\gamma_\xi m(\xi)$, for any $\gamma \in \{0,1\}^d$. If there is a constant $c > 0$ such that

$$\xi^\gamma \partial^\gamma_\xi m(\xi) \leq c, \ \xi \in \mathbb{R}^d, \ \gamma \in \{0,1\}^d,$$

then for $1 < p < \infty$ there exists a constant $B = B(p,d)$ such that $\|T_m u\|_{L^p} \leq B\|u\|_{L^p}, \ \ u \in \mathcal{S}(\mathbb{R}^d)$.

To obtain L^p-boundedness it is enough to assume that for $a(x,\xi)$ it holds that

$$|\xi^\gamma \partial_x^\lambda \partial_\xi^{\nu+\gamma} a(x,\xi)| \leq C\langle\xi\rangle^{-\varepsilon|\nu|}, \quad (x,\xi) \in \mathbb{R}^{2d},$$

for some $\varepsilon > 0$, and for all $\lambda, \nu \in \mathbb{N}_0^d$, $\gamma \in \mathbb{K}_d$.

Theorem

Let $v \in H_\Lambda^{m,q}(\mathbb{R}^d)$, $m \in \mathbb{R}, 1 < q < \infty$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then $\varphi v \in H_\Lambda^{m,q}(\mathbb{R}^d)$.

Theorem

Let $u_n \rightharpoonup 0$ *in* $L^p(\mathbb{R}^d)$ *and* $v_n \rightharpoonup 0$ *in* $H^{m,q}_\Lambda(\mathbb{R}^d)$, $m \in \mathbb{R}$. *Assume that* $\psi \in M\Gamma^m_{1/\omega,\Lambda}$. *Then, up to subsequences, there exists a distribution* $\mu_\psi \in \mathcal{S}'(\mathbb{R}^d)$ *such that for all* $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{n\to\infty} \langle u_n, \overline{T_{\bar{\psi}}(\varphi v_n)}\rangle = \langle \mu_\psi, \bar{\varphi}\rangle.$$

Theorem

Let $u_n \rightharpoonup 0$ *in* $L^p(\mathbb{R}^d)$. *Assume that*

$$\lim_{n\to\infty} \langle u_n, T_{\Lambda(x,\xi)^m}(\varphi v_n)\rangle = 0,$$

for every sequence $v_n \rightharpoonup 0$ *in* $H_\Lambda^{m,q}(\mathbb{R}^d)$, $m \in \mathbb{R}$. *Then for every* $\theta \in \mathcal{S}(\mathbb{R}^d)$, $\theta u_n \to 0$ *strongly in* $L^p(\mathbb{R}^d)$.

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for every sequence $v_n \rightharpoonup 0$ *in* $H^{m,q}_\Lambda(\mathbb{R}^d)$*,* $m \in \mathbb{R}$*. Then for every* $\theta \in \mathcal{S}(\mathbb{R}^d)$*,* $\theta u_n \to 0$ *strongly in* $L^p(\mathbb{R}^d)$*.*

Corollary

Let $u_n \rightharpoonup 0$ *in* $L^p(\mathbb{R}^d)$ *and* $a \in EM\Gamma^m_{\rho,\Lambda}$*. Assume that*

$$\lim_{n\to\infty} \langle u_n, T_a(\varphi v_n)\rangle = 0,$$

for every sequence $v_n \rightharpoonup 0$ *in* $H^{m,q}_\Lambda(\mathbb{R}^d)$*,* $m \in \mathbb{R}$*. Then for every* $\theta \in \mathcal{S}(\mathbb{R}^d)$*,* $\theta u_n \to 0$ *strongly in* $L^p(\mathbb{R}^d)$*.*

Let

$$P(x,D)u_n = \sum_{(\alpha,\beta)\in V(\mathcal{P})} x^\beta D_x^\alpha u_n = f_n, \qquad (1)$$

for some complete polyhedron $\mathcal{P}$, where $u_n \rightharpoonup 0$ in $H^{1,p}_{\mathcal{P}}$ and $\varphi f_n \to 0$ in $L^p(\mathbb{R}^d)$ for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Here $V(\mathcal{P})$ denotes the set of vertices of $\mathcal{P}$ and $p(x,\xi) = \sum_{(\alpha,\beta)\in V(\mathcal{P})} x^\beta \xi^\alpha \in M\Gamma^1_{1/\omega,\mathcal{P}}$.

Theorem

*Let $u_n \rightharpoonup 0$ in $H^{1,p}_{\mathcal{P}}(\mathbb{R}^d)$ satisfies (**??**). Then for any $v_n \rightharpoonup 0$ in $L^q(\mathbb{R}^d)$ it holds that*

$$\mu_p = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

If p is elliptic, then $\theta u_n \to 0$ in $H^{1,p}_{\mathcal{P}}$, for every $\theta \in \mathcal{S}(\mathbb{R}^d)$.

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