Classification of classical Friedrichs operators : One dimensional scalar case

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Classical Friedrichs operators

Assumptions:
$d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary;
$\mathbf{A}_k \in W^{1,\infty}(\Omega; \mathrm{M}_r(\mathbb{C}))$, $k \in \{1, \ldots, d\}$, and $\mathbf{B} \in L^\infty(\Omega; \mathrm{M}_r(\mathbb{C}))$ satisfying (a.e. on Ω):

(F1) $$\mathbf{A}_k = \mathbf{A}_k^* \,;$$

(F2) $$(\exists\, \mu_0 > 0) \quad \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \,.$$

Define $\mathcal{L}, \widetilde{\mathcal{L}} : L^2(\Omega)^r \to \mathcal{D}'(\Omega)^r$ by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u} \,, \qquad \widetilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\Big)\mathsf{u} \,.$$

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Aim: impose boundary conditions such that for any $\mathsf{f} \in L^2(\Omega)^r$ we have a unique solution of $\mathcal{L}\mathsf{u} = \mathsf{f}$.

K. O. Friedrichs: *Symmetric positive linear differential equations*, Commun. Pure Appl. Math. **11** (1958) 333–418.

$(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space ($\mathcal{H}' \equiv \mathcal{H}$), $\| \cdot \| := \sqrt{\langle \cdot \mid \cdot \rangle}$
$\mathcal{D} \subseteq \mathcal{H}$ dense subspace

Definition

Let $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

(T1) $(\forall\, \varphi, \psi \in \mathcal{D}) \qquad \langle T\varphi \mid \psi \rangle = \langle \varphi \mid \widetilde{T}\psi \rangle\,;$

(T2) $(\exists\, c > 0)(\forall\, \varphi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\varphi\| \leqslant c\|\varphi\|\,;$

(T3) $(\exists\, \mu_0 > 0)(\forall\, \varphi \in \mathcal{D}) \qquad \langle (T + \widetilde{T})\varphi \mid \varphi \rangle \geqslant \mu_0 \|\varphi\|^2\,.$

A. Ern, J.L. Guermond, G. Caplain: *An intrinsic criterion for the bijectivity of Hilbert operators related to Friedrichs' systems*, Comm. Partial Diff. Eq. **32** (2007) 317–341.

N. Antonić, K. Burazin: *Intrinsic boundary conditions for Friedrichs systems*, Comm. Partial Diff. Eq. **35** (2010) 1690–1715.

Characterisation of joint pair of abstract Friedrichs operators

$T_0 := \overline{T}$, $\widetilde{T}_0 := \overline{\widetilde{T}}$ on $\mathcal{W}_0$(closure of $\mathcal{D}$) and $T_1 := \widetilde{T}^*$, $\widetilde{T}_1 := T^*$ on $\mathcal{W}$(the graph space).
Boundary map (form): $D : \mathcal{W} \to \mathcal{W}'$,

$$[u \,|\, v] := {}_{\mathcal{W}'}\langle Du, v\rangle_{\mathcal{W}} := \langle\, T_1 u \mid v \,\rangle - \langle\, u \mid \widetilde{T}_1 v \,\rangle \,.$$

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$(\mathcal{W}, [\cdot \,|\, \cdot])$ is an *indefinite inner product space*. $\quad$ ($\exists\, 0 \neq u \in \mathcal{W}, \quad [u \,|\, u] = 0$)

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For $\mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W}$ we introduce two conditions:

(V1)
$$(\forall\, u \in \mathcal{V}) \qquad [u \,|\, u] \geqslant 0\,,$$
$$(\forall\, v \in \widetilde{\mathcal{V}}) \qquad [v \,|\, v] \leqslant 0\,.$$

(V2)
$$\mathcal{V}^{[\perp]} = \widetilde{\mathcal{V}}\,,\ \widetilde{\mathcal{V}}^{[\perp]} = \mathcal{V}\,.$$

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Well-posedness:

Theorem (Ern, Guermond, Caplain, 2007)

(T1)–(T3) + (V1)–(V2) $\implies T_1|_{\mathcal{V}}, \widetilde{T}_1|_{\widetilde{\mathcal{V}}}$ *bijective realisations .*

We seek for bijective closed operators $S \equiv \widetilde{T}^*|_{\mathcal{V}}$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^*\,,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$. We call (S, S^*) an adjoint pair of bijective realisations relative to $(T, \widetilde{T})$.

Existance, multiplicity and classification

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Theorem (Antonić, Erceg, Michelangeli, 2017)

Let $(T, \widetilde{T})$ satisfies (T1)–(T3).

(i) *There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.*

(ii)

$\ker \widetilde{T}^* \neq \{0\} \ \& \ \ker T^* \neq \{0\} \implies$ *uncountably many adjoint pairs of bijective realisations with signed boundary map*

$\ker \widetilde{T}^* = \{0\}$ *or* $\ker T^* = \{0\} \implies$ *only one adjoint pair of bijective realisations with signed boundary map*

Classification 1/2

Let $(T_0, \widetilde{T}_0)$ and $(T_1, \widetilde{T}_1)$ be two pairs of mutually adjoint, closed and densely defined operators on $\mathcal{H}$ satisfying

$$T_0 \subseteq (\widetilde{T}_0)^* = T_1 \qquad \text{and} \qquad \widetilde{T}_0 \subseteq (T_0)^* = \widetilde{T}_1 \,,$$

which admit a pair $(T_{\mathrm{r}}, T_{\mathrm{r}}^*)$ of reference operators that are closed, satisfy $T_0 \subseteq T_{\mathrm{r}} \subseteq T_1$, equivalently $\widetilde{T}_0 \subseteq T_{\mathrm{r}}^* \subseteq \widetilde{T}_1$, and are invertible with everywhere defined bounded inverses T_{r}^{-1} and $(T_{\mathrm{r}}^*)^{-1}$.

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Then

(i)

$$\operatorname{dom} T_1 = \operatorname{dom} T_\mathrm{r} \dot{+} \ker T_1 \qquad \text{and} \qquad \operatorname{dom} \widetilde{T}_1 = \operatorname{dom} T_\mathrm{r}^* \dot{+} \ker \widetilde{T}_1 \, .$$

Let $(T_0, \widetilde{T}_0)$ and $(T_1, \widetilde{T}_1)$ be two pairs of mutually adjoint, closed and densely defined operators on $\mathcal{H}$ satisfying

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Then

(i)

$$\operatorname{dom} T_1 = \operatorname{dom} T_{\rm r} \dotplus \ker T_1 \qquad \text{and} \qquad \operatorname{dom} \widetilde{T}_1 = \operatorname{dom} T_{\rm r}^* \dotplus \ker \widetilde{T}_1 \,.$$

The corresponding (non-orthogonal) projections

$$\begin{aligned} p_{\rm r} &: \operatorname{dom} T_1 \to \operatorname{dom} T_{\rm r} \,, \qquad & p_{\tilde{\rm r}} &: \operatorname{dom} \widetilde{T}_1 \to \operatorname{dom} T_{\rm r}^* \,, \\ p_{\rm k} &: \operatorname{dom} T_1 \to \ker T_1 \,, \qquad & p_{\tilde{\rm k}} &: \operatorname{dom} \widetilde{T}_1 \to \ker \widetilde{T}_1 \,, \end{aligned}$$

satisfying

$$\begin{aligned} p_{\rm r} &= T_{\rm r}^{-1} T_1 \,, \qquad & p_{\tilde{\rm r}} &= (T_{\rm r}^*)^{-1} \widetilde{T}_1 \,, \\ p_{\rm k} &= \mathbb{1} - p_{\rm r} \,, \qquad & p_{\tilde{\rm k}} &= \mathbb{1} - p_{\tilde{\rm r}} \,, \end{aligned}$$

and being continuous with respect to the graph norms.

Classification 2/2

(ii)

$$\Big\{(A, A^*) : \widetilde{T}_0 \subseteq A \subseteq T_1\Big\} \xleftrightarrow{1:1} \Big\{(B, B^*) : B : \mathcal{Z} \to \widetilde{\mathcal{Z}} \text{ closed densely defined}\Big\},$$

where $\mathcal{Z}, \widetilde{\mathcal{Z}}$ run through closed subspaces of $\ker T_1$ and $\ker \widetilde{T}_1$ respectively.

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The correspondence is given by

$$\begin{aligned}
\operatorname{dom} A \;&=\; \Big\{u \in \operatorname{dom} T_1 \;:\; p_{\mathrm{k}}u \in \operatorname{dom} B\,,\; P_{\widetilde{\mathcal{Z}}}(T_1 u) = B(p_{\mathrm{k}}u)\Big\}\,,\\
\operatorname{dom} A^* \;&=\; \Big\{v \in \operatorname{dom} \widetilde{T}_1 \;:\; p_{\tilde{\mathrm{k}}}v \in \operatorname{dom} B^*\,,\; P_{\mathcal{Z}}(\widetilde{T}_1 v) = B^*(p_{\tilde{\mathrm{k}}}v)\Big\}\,.
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$$\Big\{(A,A^*) : \widetilde{T}_0 \subseteq A \subseteq T_1\Big\} \overset{1:1}{\longleftrightarrow} \Big\{(B,B^*) : B : \mathcal{Z} \to \widetilde{\mathcal{Z}} \text{ closed densely defined}\Big\},$$

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Conversely, by

$$\begin{aligned} \operatorname{dom} B &= p_{\mathrm{k}} \operatorname{dom} A\,, & \mathcal{Z} &= \overline{\operatorname{dom} B}\,, & B(p_{\mathrm{k}} u) &= P_{\widetilde{\mathcal{Z}}}(T_1 u)\,, \\ \operatorname{dom} B^* &= p_{\tilde{\mathrm{k}}} \operatorname{dom} A^*\,, & \widetilde{\mathcal{Z}} &= \overline{\operatorname{dom} B^*}\,, & B^*(p_{\tilde{\mathrm{k}}} v) &= P_{\mathcal{Z}}(\widetilde{T}_1 v)\,, \end{aligned}$$

where $P_{\mathcal{Z}}$ and $P_{\widetilde{\mathcal{Z}}}$ are the *orthogonal* projections from $\mathcal{H}$ onto $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$.

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$$\begin{array}{rclcrclcrcl} \operatorname{dom} B &=& p_{\mathrm{k}} \operatorname{dom} A\,, & & \mathcal{Z} &=& \overline{\operatorname{dom} B}\,, & & B(p_{\mathrm{k}} u) &=& P_{\widetilde{\mathcal{Z}}}(T_1 u)\,, \\ \operatorname{dom} B^* &=& p_{\tilde{\mathrm{k}}} \operatorname{dom} A^*\,, & & \widetilde{\mathcal{Z}} &=& \overline{\operatorname{dom} B^*}\,, & & B^*(p_{\tilde{\mathrm{k}}} v) &=& P_{\mathcal{Z}}(\widetilde{T}_1 v)\,, \end{array}$$

where $P_{\mathcal{Z}}$ and $P_{\widetilde{\mathcal{Z}}}$ are the *orthogonal* projections from $\mathcal{H}$ onto $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$.

G. Grubb: *A characterization of the non-local boundary value problems associated with an elliptic operator*, Ann. Scuola Norm. Sup. Pisa **22** (1968) 425–513.

N. Antonić, M.Erceg, A. Michelangeli: *Friedrichs systems in a Hilbert space framework: solvability and multiplicity*, J. Differential Equations 263 (2017) 8264-8294.

One-dimensional scalar case: Preliminaries 1/5

Theorem

$(T_0, \widetilde{T}_0)$ *is a joint pair of closed abstract Friedrichs operators then*

$$\mathcal{W} = \mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1 .$$

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$\Omega = (a, b)$, $a < b$, $\mathcal{D} = C_c^\infty(a, b)$ and $\mathcal{H} = L^2(a, b)$. $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$:

$$T\varphi := (\alpha\varphi)' + \beta\varphi \qquad \text{and} \qquad \widetilde{T}\varphi := -(\alpha\varphi)' + (\overline{\beta} + \alpha')\varphi \ .$$

Here $\alpha \in W^{1,\infty}((a, b); \mathbb{R})$, $\beta \in L^\infty((a, b); \mathbb{C})$ and for some $\mu_0 > 0$, $2\Re\beta + \alpha' \geq 2\mu_0 > 0$.

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The graph space :

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Equivalently ,

$$u \in \mathcal{W} \iff \alpha u \in H^1(a,b) \, .$$

So, by Sobolev embedding $\alpha u \in C(a,b)$. Implies the evaluation $(\alpha u)(x)$ is well defined. However, u is not necessarily continuous so $\alpha(x)u(x)$ is not meaningful.

Lemma

Let $I := [a,b] \setminus \alpha^{-1}(\{0\})$. *Then* $\mathcal{W} \subseteq H^1_{\rm loc}(I)$, *i.e. for any* $u \in \mathcal{W}$ *and* $[c,d] \subseteq I$, $c < d$, *we have* $u|_{[c,d]} \in H^1(c,d)$.

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Proof : Since α is continuous, I is relatively open in $[a,b]$. Let us take $[c,d] \subseteq I$, $c < d$, define $\alpha_0 := \min_{x\in[c,d]} |\alpha(x)|$. Let $u \in C_c^\infty(\mathbb{R})$, then

$$\begin{aligned}\|u'\|_{L^2(c,d)} &= \Big\|\frac{1}{\alpha}\alpha u'\Big\|_{L^2(c,d)} \leq \frac{1}{\alpha_0}\|\alpha u'\|_{L^2(c,d)} \\ &= \frac{1}{\alpha_0}\|(\alpha u)' - \alpha' u\|_{L^2(c,d)} \leq \frac{1}{\alpha_0}\Big(\|(\alpha u)'\|_{L^2(c,d)} + \|\alpha' u\|_{L^2(c,d)}\Big) \\ &\leq \frac{1}{\alpha_0}\Big(\|(\alpha u)'\| + \|\alpha'\|_{L^\infty(a,b)}\|u\|\Big) \leq \frac{1+\|\alpha\|_{W^{1,\infty}(a,b)}}{\alpha_0}\|u\|_{\mathcal{W}} \,.\end{aligned}$$

By density of $C_c^\infty(\mathbb{R})$ in $\mathcal{W}$ we get $u|_{[c,d]} \in H^1(c,d)$ and there exists $C > 0$ (dependent on c, d) :

$$\|u\|_{H^1(c,d)} \leq C\|u\|_{\mathcal{W}} \,, \quad u \in \mathcal{W} \,.$$

The boundary operator can be written explicitly as

$$ {}_{\mathcal{W}'}\langle\, Du, v \,\rangle_{\mathcal{W}} = \big(\alpha u\overline{v}\big)(b) - \big(\alpha u\overline{v}\big)(a)\;, \quad u, v \in \mathcal{W}\;, $$

where we define

$$ \big(\alpha u\overline{v}\big)(x) := \left\{ \begin{array}{lcl} 0 & , & \alpha(x) = 0 \\ \alpha(x)u(x)\overline{v(x)} & , & \alpha(x) \neq 0 \end{array} \right. \quad , \quad x \in [a, b]\,. $$

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The domain of the closures T_0 and $\widetilde{T}_0$ satisfies $\mathcal{W}_0 = \mathrm{cl}_{\mathcal{W}} C_c^\infty(\mathbb{R})$, is characterised as

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$$ \mathcal{W}_0 = \Big\{ u \in \mathcal{W} : (\alpha u)(a) = (\alpha u)(b) = 0 \Big\}\,. $$

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$$\mathcal{W}_0 = \Big\{ u \in \mathcal{W} : (\alpha u)(a) = (\alpha u)(b) = 0 \Big\}\,.$$

Proof : Since $\ker D = \mathcal{W}_0$, so it is sufficient to prove that this set is $\ker D$. Let $u \in \mathcal{W}$ such that $(\alpha u)(a) = (\alpha u)(b) = 0$, then

$$\forall v \in \mathcal{W}\,, \quad {}_{\mathcal{W}'}\langle\, Du, v\,\rangle_{\mathcal{W}} = \left(\alpha u \overline{v}\right)(b) - \left(\alpha u \overline{v}\right)(a) = 0 - 0 = 0\,.$$

So, $\{u \in \mathcal{W} : (\alpha u)(a) = (\alpha u)(b) = 0\} \subseteq \ker D$.

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$$\forall v \in \mathcal{W}\,, \quad _{\mathcal{W}'}\langle\, Du, v\,\rangle_{\mathcal{W}} = (\alpha u\overline{v})\,(b) - (\alpha u\overline{v})\,(a) = 0 - 0 = 0\,.$$

So, $\{u \in \mathcal{W} : (\alpha u)(a) = (\alpha u)(b) = 0\} \subseteq \ker D$. Conversely, let $u \in \ker D \subset \mathcal{W}$ then for any $v \in H^1(a, b) \subseteq \mathcal{W}$,

$$0 = {_{\mathcal{W}'}\langle\, Du, v\,\rangle_{\mathcal{W}}} = (\alpha u\overline{v})(b) - (\alpha u\overline{v})(a) = (\alpha u)(b)\overline{v(b)} - (\alpha u)(a)\overline{v(a)}\,.$$

Here, v was continuous (Sobolev embedding). So, $(\alpha u)(b) = 0, (\alpha u)(a) = 0$. Hence, $\ker D \subseteq \{u \in \mathcal{W} : (\alpha u)(a) = (\alpha u)(b) = 0\}$.

Lemma

$$\dim(\mathcal{W}/\mathcal{W}_0) = \begin{cases} 2 & , \quad \alpha(a)\alpha(b) \neq 0 \,, \\ 1 & , \quad \big(\alpha(a) = 0 \wedge \alpha(b) \neq 0\big) \vee \big(\alpha(a) \neq 0 \wedge \alpha(b) = 0\big) \,, \\ 0 & , \quad \alpha(a) = \alpha(b) = 0 \,. \end{cases}$$

Proof : If $\alpha(a)\alpha(b) \neq 0$, then choose $\varphi, \psi \in \mathcal{W}$, such that $\varphi(a) = 1$, $\varphi(b) = 0$ and $\psi(a) = 0$, $\psi(b) = 1$. Define $\hat{\varphi} := \varphi + \mathcal{W}_0$ and $\hat{\psi} := \psi + \mathcal{W}_0$. Then $E := \{\hat{\varphi}, \hat{\psi}\}$ is a basis of $\mathcal{W}/\mathcal{W}_0$.

If E were linearly dependent then for some non-zero scalar r we would have $\hat{\psi} = r\hat{\varphi}$, implying $\hat{\psi} - r\hat{\varphi} = \hat{0} = \mathcal{W}_0$. Hence, $\psi - r\varphi \in \mathcal{W}_0$, so

$$\big(\alpha(\psi - r\varphi)\big)(a) = \big(\alpha(\psi - r\varphi)\big)(b) = 0 \,.$$

But,

$$\big(\alpha(\psi - r\varphi)\big)(a) = \alpha(a)\psi(a) - r\alpha(a)\varphi(a) = 0 - r\alpha(a) = -r\alpha(a) \neq 0 \,,$$

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Now let $u \in \mathcal{W}$, then

$$u - u(a)\varphi - u(b)\psi \in \mathcal{W}_0 \,,$$

means E spans $\mathcal{W}/\mathcal{W}_0$. So, E is a basis of $\mathcal{W}/\mathcal{W}_0$, hence $\dim(\mathcal{W}/\mathcal{W}_0) = 2$.

If $\alpha(a) = 0$ and $\alpha(b) \neq 0$, then we take $\varphi \in \mathcal{W}$ such that $\varphi(b) = 1$. Since,

$$u \in \mathcal{W} \implies u - u(b)\varphi \in \mathcal{W}_0 \ ,$$

we get $\mathrm{span}\{\varphi + \mathcal{W}_0\} = \mathcal{W}/\mathcal{W}_0$. Hence, $\dim(\mathcal{W}/\mathcal{W}_0) = 1$.
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i) If $\min_{x\in[a,b]} |\alpha(x)| > \alpha_0 > 0$, then $\dim\big(H^1(a,b)/H^1_0(a,b)\big) = 2$.

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Thus, when $\alpha(a)\alpha(b) = 0$ there is only one bijective realisation of T_0. When case $\alpha(a)\alpha(b) \neq 0$ there are infinitely many bijective realisations if and only if $\dim(\ker T_1) = \dim(\ker \widetilde{T}_1)$.

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We shall justify and improve these conclusions by a direct inspection.

Case 1: $\alpha(a)\alpha(b) = 0$ **:** For $\alpha(a) = \alpha(b) = 0$, We have $D \equiv 0$. So, $\mathcal{W}_0 = \ker(D) = \mathcal{W}$, thus the only possible choice is $(\mathcal{V}, \widetilde{\mathcal{V}}) = (\mathcal{W}, \mathcal{W})$. Hence, the only possible pair of mutually adjoint bijective realisation is $(T_1, \widetilde{T}_1)$.

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For $\alpha(a) = 0, \alpha(b) > 0$, We have :

$$\forall u, v \in \mathcal{W}\,, \quad {}_{\mathcal{W}'}\langle\, Du, v\,\rangle_{\mathcal{W}} = \alpha(b)u(b)\overline{v(b)}\;.$$
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So, pair $(\mathcal{W}, \mathcal{W}_0)$ satisfies condition (V1). Furthermore, $(T_1|_{\mathcal{W}}, \widetilde{T}_1|_{\mathcal{W}_0}) = (T_1, \widetilde{T}_0)$ is trivially a pair of mutually adjoint operators and so it is a pair of mutually adjoint bijective realisations relative to $(T, \widetilde{T})$. Since this implies that $\ker T_1 = \{0\}$, $(T_1, \widetilde{T}_0)$ is the only pair of mutually adjoint bijective realisations relative to $(T, \widetilde{T})$.

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Similarly, for $\alpha(a) = 0, \alpha(b) < 0$, we have $(\mathcal{V}, \widetilde{\mathcal{V}}) = (\mathcal{W}_0, \mathcal{W})$.

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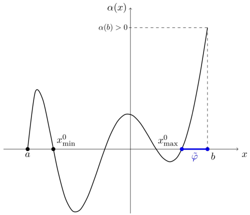

Case 2: $\alpha(a)\alpha(b) < 0$: $\mathcal{W}_0 = \{u \in \mathcal{W} : u(a) = u(b) = 0\}$
For $\alpha(a) > 0$ and $\alpha(b) < 0$. Then for any $u \in \mathcal{W}$ we have

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Hence, we get $(T_0, \widetilde{T}_1) = (T_1|_{\mathcal{W}_0}, \widetilde{T}_1|_{\mathcal{W}})$ is the only pair of mutually adjoint bijective realisations relative to $(T, \widetilde{T})$.

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Kernels : Although in this case $\dim(\ker T_1) + \dim(\ker \widetilde{T}_1) = 2$, we have only one bijective realisation. So, for $\alpha(a) > 0$ we have $(\dim(\ker T_1), \dim(\ker \widetilde{T}_1)) = (2, 0)$, while for $\alpha(a) < 0$ it is $(\dim(\ker T_1), \dim(\ker \widetilde{T}_1)) = (0, 2)$.

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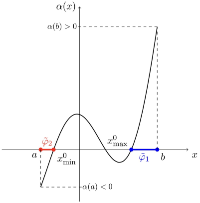
Let $\alpha(a) < 0$. We have $\alpha^{-1}(\{0\}) \neq \emptyset$.



Case 3: $\alpha(a)\alpha(b) > 0$: $\mathcal{W}_0 = \{u \in \mathcal{W} : u(a) = u(b) = 0\}$, the boundary operator is

$$_{\mathcal{W}'}\langle\, Du, v \,\rangle_{\mathcal{W}} = \alpha(b)u(b)\overline{v(b)} - \alpha(a)u(a)\overline{v(a)}\ , \quad u, v \in \mathcal{W}\ .$$

Let us define

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$$\mathcal{V} := \left\{ u \in \mathcal{W} : u(b) = \sqrt{\frac{\alpha(a)}{\alpha(b)}}\,u(a) \right\}.$$

For an arbitrary $u \in \mathcal{V}$ and $v \in \mathcal{W}$ we have

$$\begin{aligned} _{\mathcal{W}'}\langle\, Du, v\,\rangle_{\mathcal{W}} &= \alpha(b)u(b)\overline{v(b)} - \alpha(a)u(a)\overline{v(a)} \\ &= \alpha(b)\left(u(b)\overline{v(b)} - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\,u(a)\sqrt{\frac{\alpha(a)}{\alpha(b)}}\,\overline{v(a)} \right) \\ &= \alpha(b)u(b)\overline{\left(v(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\,v(a) \right)}\,. \end{aligned}$$

In particular,

$$(\forall u, v \in \mathcal{V}) \qquad {}_{\mathcal{W}'}\langle\, Du, v\,\rangle_{\mathcal{W}} = 0\,,$$

implying that $(\mathcal{V}, \mathcal{V})$ satisfies condition (V1) and that $\mathcal{V} \subseteq \mathcal{V}^{[\perp]}$.

Now let $v \in \mathcal{V}^{[\perp]}$. Then for any $u \in \mathcal{V}$

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Therefore, $(T_{\mathrm{r}}, T_{\mathrm{r}}^*)$ is indeed a mutually adjoint pair of bijective realisations relative to $(T, \widetilde{T})$. It is evident that $\mathcal{W}_0 \subsetneqq \mathcal{V} \subsetneqq \mathcal{W}$, hence there are infinitely many bijective realisations.

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Therefore, $(T_{\mathrm{r}}, T_{\mathrm{r}}^*)$ is indeed a mutually adjoint pair of bijective realisations relative to $(T, \widetilde{T})$. It is evident that $\mathcal{W}_0 \subsetneqq \mathcal{V} \subsetneqq \mathcal{W}$, hence there are infinitely many bijective realisations.

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From the classification theory we have $u \in \mathcal{W}$ belongs to $\operatorname{dom} T_{c,d}$ if and only if

$$P_{\ker \widetilde{T}_1}(T_1 u) = B(p_{\mathrm{k}} u)$$

Non-orthogonal projections : For any $u \in \mathcal{W}$ there exist unique $u_{\mathrm{r}} \in \mathcal{V}$ and $u_{\mathrm{k}} \in \ker T_1$ such that $u = u_{\mathrm{r}} + u_{\mathrm{k}}$. Moreover, u_{k} is of the form $C_u\varphi$, so using

$$u(a) = u_{\mathrm{r}}(a) + C_u\varphi(a)$$
$$u(b) = u_{\mathrm{r}}(b) + C_u\varphi(b)$$

and $u_{\mathrm{r}}(b) = \sqrt{\frac{\alpha(a)}{\alpha(b)}}u_{\mathrm{r}}(a)$. We get

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$$\tilde{C}_u = \frac{u(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}} u(a)}{\tilde{\varphi}(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}} \tilde{\varphi}(a)} \,.$$

Orthogonal projection :

$$P_{\ker \widetilde{T}_1}(T_1 u) = \frac{1}{\|\tilde{\varphi}\|^2}\langle\, T_1 u \mid \tilde{\varphi}\,\rangle\tilde{\varphi} = \frac{1}{\|\tilde{\varphi}\|^2}\,{}_{\mathcal{W}'}\langle\, Du, \tilde{\varphi}\,\rangle_{\mathcal{W}}\tilde{\varphi}$$

$$= \frac{1}{\|\tilde{\varphi}\|^2}\Big(\alpha(b)u(b)\overline{\tilde{\varphi}(b)} - \alpha(a)u(a)\overline{\tilde{\varphi}(a)}\Big)\tilde{\varphi}\;.$$

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Which gives $u \in \mathcal{W}$ belongs to $\operatorname{dom} T_{c,d}$ if and only if

[1] $$\left(\frac{\alpha(b)\overline{\tilde{\varphi}(b)}}{\|\tilde{\varphi}\|^2} - \frac{(c+id)}{\varphi(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)}\right)u(b) = \left(\frac{\alpha(a)\overline{\tilde{\varphi}(a)}}{\|\tilde{\varphi}\|^2} - \frac{(c+id)\sqrt{\frac{\alpha(a)}{\alpha(b)}}}{\varphi(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)}\right)u(a)\ .$$

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Similarly, $u \in \mathcal{W}$ is in $\operatorname{dom} T^*_{c,d}$ if and only if

$$[2]\ \left(\alpha(b)\overline{\varphi(b)} - \frac{\|\tilde{\varphi}\|^2(c-id)}{\tilde{\varphi}(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\tilde{\varphi}(a)}\right)u(b) = \left(\alpha(a)\overline{\varphi(a)} - \frac{\|\tilde{\varphi}\|^2(c-id)\sqrt{\frac{\alpha(a)}{\alpha(b)}}}{\tilde{\varphi}(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\tilde{\varphi}(a)}\right)u(a)\ .$$

Classification 8/8

So, the set of all pairs of mutually adjoint bijective realisations relative to $(T, \widetilde{T})$ is given by

$$[3] \qquad \Big\{(T_{c,d}, T^*_{c,d}) : c, d \in \mathbb{R}^2 \setminus \{(0,0)\}\Big\} \bigcup \big\{(T_{\mathrm{r}}, T^*_{\mathrm{r}})\big\} \, .$$

Kernels : If $\min_{x\in[a,b]} |\alpha(x)| > 0$, then simply

$$\varphi(x) = \frac{1}{\alpha(x)} \exp\Big(-\int \frac{\beta(x)}{\alpha(x)}\, dx\Big) \qquad \text{and} \qquad \tilde{\varphi}(x) = \exp\Big(\int \frac{\overline{\beta(x)}}{\alpha(x)}\, dx\Big) \, .$$

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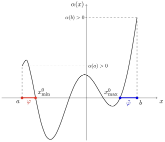
If $\alpha^{-1}(\{0\}) \cap (a,b) \neq \emptyset$,



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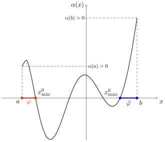

Summary :

α at end-points	No. of bij. realisations	$(\mathcal{V}, \widetilde{\mathcal{V}})$	
$\alpha(a)\alpha(b) \leq 0$	1	$\alpha(a) \geq 0 \wedge \alpha(b) \leq 0$	$(\mathcal{W}_0, \mathcal{W})$
		$\alpha(a) \leq 0 \wedge \alpha(b) \geq 0$	$(\mathcal{W}, \mathcal{W}_0)$
$\alpha(a)\alpha(b) > 0$	∞	[3] (see [1] and [2])	

Examples

Example 1 : Take the interval $\Omega := (0,2)$ and coefficients $\alpha(x) = 1 - x$ and $\beta = 1$. Then

$$T\varphi = ((1-x)\varphi)' + \varphi$$

and

$$\widetilde{T}\varphi = -((1-x)\varphi)' \, .$$

Here $2\Re\beta + \alpha' = 2 - 1 = 1 > 0$ on $(0,2)$, meaning that $(T, \widetilde{T})$ is a pair of abstract Friedrichs operators.

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Solving on $(0, 1)$ and $(1, 2)$ separately we get that for $\varphi \in \ker T_1$ necessarily

$$\varphi = \begin{cases} c_1 & , \quad \text{in } (0, 1) \\ c_2 & , \quad \text{in } (1, 2) \, , \end{cases}$$

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for some constants $c_1, c_2 \in \mathbb{C}$. We have $\varphi \in \mathcal{W}$. Indeed, it is evident that $\varphi \in L^2(0,2)$, while for $\psi \in C_c^\infty(0,2)$ we have

$$\begin{aligned} \int_0^2 (1-x)\varphi(x)\psi'(x)\,dx &= \int_0^1 (1-x)\varphi(x)\psi'(x)\,dx + \int_1^2 (1-x)\varphi(x)\psi'(x)\,dx \\ &= c_1 \int_0^1 (1-x)\psi'(x)\,dx + c_2 \int_1^2 (1-x)\psi'(x)\,dx \\ &= c_1 \int_0^1 \psi(x)\,dx + c_2 \int_1^2 \psi(x)\,dx = \int_0^2 \varphi(x)\psi(x)\,dx \, . \end{aligned}$$

This means $((1-x)\varphi)' = -\varphi \in L^2(0,2)$, thus $\varphi \in \mathcal{W}$. Therefore, $\dim\ker T_1 = 2$ (since we have two parameters in the definition of φ).

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$$\tilde{\varphi}(x) = \begin{cases} \frac{d_1}{1-x} & , \quad x \in (0,1) \\ \frac{d_2}{1-x} & , \quad x \in (1,2) \end{cases},$$

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It is interesting to note that for $c_1 \neq c_2$ we have $\varphi' \notin L^2(0,2)$, because $\varphi' = (c_2 - c_1)\delta_1$ (here δ_1 is the Dirac measure at 1) and so $\varphi \notin H^1(0,2)$. Thus, $H^1(0,2) \subsetneqq \mathcal{W}$.

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Moreover, $\tilde{\varphi} \in H^1_{\rm loc}([0,2] \setminus \{1\})$ for any choice of parameters d_1, d_2. Indeed, for any subinterval $[c,d] \subseteq [0,2] \setminus \{1\}$ we have $\tilde{\varphi}|_{(c,d)} \in \mathrm{H}^1(c,d)$. Since $\tilde{\varphi} \notin \mathcal{W}$ this shows that $\mathcal{W}$ is indeed a proper subspace of $H^1_{\rm loc}([0,2] \setminus \{1\})$, i.e. $\mathcal{W} \subsetneqq H^1_{\rm loc}([0,2] \setminus \{1\})$.

...thank you for your attention :)