Singular solution of the Hartree equation with a delta potential

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Brijuni Applied Mathematics Workshop 2021

MiTPDE

HrZZ Croatian Science Foundation

- We consider the Cauchy problem for the Hartree equation with the delta potential

$$\begin{aligned} iu_t + \triangle u &= (w * |u|^2)u + \delta u, \\ u(0,x) &= a(x), \end{aligned} \tag{1}$$

where $u = u(t,x),\ t \in [0,T),\ x \in \mathbb{R}^3$, $w : \mathbb{R}^3 \to \mathbb{R}$ is a given measurable function and δ denotes the Dirac delta distribution.

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- Well-posedness for (1) holds in the setting of the fractional (singular) Sobolev spaces $H^2_\alpha(\mathbb{R}^3)$:

$$\begin{aligned} iu_t + \triangle_\alpha u &= (w * |u|^2)u, \\ u(0,x) &= a(x), \end{aligned} \quad (2)$$

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- Well-posedness for (1) holds in the setting of the fractional (singular) Sobolev spaces $H^2_\alpha(\mathbb{R}^3)$:

$$\begin{aligned} iu_t + \triangle_\alpha u &= (w * |u|^2)u, \\ u(0,x) &= a(x), \end{aligned} \tag{2}$$

where $\triangle_\alpha$ is the fractional Laplacian.
- We will see that (1) is well-posed in a H^2-based Colombeau algebra, in which some classical spaces are embedded and then we will connect Colombeau solution and solution of (2).

- Colombeau–type algebras are motivated by nonlinearities and singularities appearing in (1). We embed the delta distribution in algebra of this type, and then the product δu obtains a meaning.
- The space of distributions is embedded by taking a convolution with a mollifier: $u \mapsto u * \rho_\varepsilon = u_\varepsilon$. Hence we consider a regularized equation

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon - (w * |u_\varepsilon|^2)u_\varepsilon &= \phi_\varepsilon u_\varepsilon \\ u_\varepsilon(0,x) &= a_\varepsilon(x), \end{aligned} \tag{3}$$

- Regularization offers the possibility to examine convergence of the net of solutions and we use it to connect the Colombeau solution with the solution of (2).

- Notation: $\mathcal{D}(\mathbb{R}^3)$ and $\mathcal{S}(\mathbb{R}^3)$ are spaces of smooth functions with compact support and rapidly decreasing functions, respectively. We denote by $W^{m,p}(\mathbb{R}^3)$, $1 \le p \le \infty$, the usual Sobolev space. When $p = 2$ we use standard notation $W^{m,2}(\mathbb{R}^3) = H^m(\mathbb{R}^3)$ and $H^\infty(\mathbb{R}^3) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^3)$.
- Let $T > 0$. We define $\mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ the vector space of nets $(u_\varepsilon)_\varepsilon$ of functions $u_\varepsilon \in C([0,T), H^2(\mathbb{R}^3)) \cap C^1([0,T), L^2(\mathbb{R}^3))$, $\varepsilon \in (0,1)$, with the property that there exists $N \in \mathbb{N}$ such that

$$\max\{\sup_{t\in[0,T)} \|u_\varepsilon(t)\|_{H^2}, \sup_{t\in[0,T)} \|\partial_t u_\varepsilon(t)\|_2\} = \mathcal{O}(\varepsilon^{-N}),\ \varepsilon \to 0.$$

- We define $\mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ as the vector space of nets $(u_\varepsilon)_\varepsilon$ such that for every $M \in \mathbb{N}$ it holds that

$$\max\{\sup_{t\in[0,T)} \|u_\varepsilon(t)\|_{H^2}, \sup_{t\in[0,T)} \|\partial_t u_\varepsilon(t)\|_2\} = \mathcal{O}(\varepsilon^{M}),\ \varepsilon \to 0, \quad \text{respectively.}$$

- We call elements of $\mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ moderate functions and elements of $\mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ negligible functions. The quotient space

$$\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3) = \mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)/\mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$$

is a Colombeau type vector space. This is a multiplicative algebra, since $H^2(\mathbb{R}^3)$ itself is an algebra.

The space $\mathcal{G}_{H^2}(\mathbb{R}^3) = \mathcal{E}_{H^2}(\mathbb{R}^3)/\mathcal{N}_{H^2}(\mathbb{R}^3)$ is defined in a similar manner, but with representatives independent in the time variable t. This space is also an algebra.

- The basic operations of addition, multiplication and differentiation are done component–wise, that is

$$u + v = [(u_\varepsilon + v_\varepsilon)_\varepsilon], \quad u \cdot v = [(u_\varepsilon \cdot v_\varepsilon)_\varepsilon], \quad \partial^\alpha u = [(\partial^\alpha u_\varepsilon)_\varepsilon].$$

- We define differentiation in this algebra, although it is not a closed operation. If $u \in \mathcal{G}_{C^1,H^2}$, then $\partial_x^\alpha u$, $|\alpha| \leq 2$ is represented by $\partial_x^\alpha u_\varepsilon$ which has moderate growth in $L^2(\mathbb{R}^3)$ and therefore gives rise to an element of a quotient vector space $\mathcal{G}_{C,L^2}$, defined analogously as $\mathcal{G}_{C^1,H^2}$ - with the difference that representatives have bounded growth only in L^2–norm, for any $t \in [0, T)$. We will see that the equation (1) has sense in $\mathcal{G}_{C,L^2}$. Also it is easily seen that $\mathcal{G}_{C^1,H^2} \subset \mathcal{G}_{C,L^2}$.

Next we show how to embed spaces of distributions in $\mathcal{G}_{C^1,H^2}$.
For that purpose we define a mollifier ρ_ε such that $\rho_\varepsilon(x) = \varepsilon^{-3}\rho(\frac{x}{\varepsilon})$, where $\rho \in S(\mathbb{R}^3)$ satisfies conditions

$$\int \rho(x)dx = 1,$$
$$\int x^\alpha \rho(x)dx = 0, \quad \text{for all } |\alpha| \geq 1.$$

The delta function can be embedded in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ by convolution with ρ_ε. Actually, $\delta * \rho_\varepsilon = \rho_\varepsilon$ so ρ_ε itself is a representative of the delta function. We will prove that in this algebra, one more representative of the delta function is given by a *strict delta net*, defined as follows.

Definition

A strict delta net $\phi_\varepsilon \in \mathcal{D}(\mathbb{R}^3)$ is a net satisfying

i) $\mathrm{supp}(\phi_\varepsilon) \to \{0\},\ \varepsilon \to 0$,
ii) $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \phi_\varepsilon(x)dx = 1$,
iii) $\int |\phi_\varepsilon(x)|dx$ is bounded uniformly in ε.

A strict delta net can be defined using ρ_ε as $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$, where χ is a cut–off function. Specifically, $\chi \in \mathcal{D}(\mathbb{R}^3)$, $0 \le \chi \le 1$ and $\chi = 1$ on $B_1(0)$ (unit ball with center at the origin) and supp $\chi \subset B_2(0)$.

Theorem

There exists a strict delta net ϕ_ε such that the difference $\rho_\varepsilon - \phi_\varepsilon$ belongs to $\mathcal{N}_{H^2}(\mathbb{R}^3)$ and both ρ_ε and ϕ_ε are representatives for the embedded delta function $[(\rho_\varepsilon)_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$.

The product of the embedded delta function in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ and an element of $\mathcal{G}_{C^1,H^2}$ belongs to $\mathcal{G}_{C^1,H^2}$. This is explained by the next theorem.

Theorem

Let $u \in \mathcal{G}_{C^1,H^2}$ and $(\rho_\varepsilon)_\varepsilon$ is the representative of δ in $\mathcal{G}_{H^2}(\mathbb{R}^3)$. Then $u \cdot [(\rho_\varepsilon)] \in \mathcal{G}_{C^1,H^2}$.

The next theorem explains embedding of functions in the space $\mathcal{G}_{C^1,H^2}$.

Theorem

Define the mapping $\iota : W^{1,\infty}([0,T), L^2(\mathbb{R}^3)) \to \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ *by*

$$\iota(u) = [(u_\varepsilon)_\varepsilon]$$

where

$$u_\varepsilon(t,x) = \int_{\mathbb{R}^3} u(t,y)\rho_\varepsilon(x-y)dy \quad \text{for any } t \in [0,T). \tag{4}$$

(i) *The mapping* ι *is a linear injection. Restriction of the derivative* ∂^α*, for any* $\alpha \in \mathbb{N}^{1+3}$*, from* $\mathcal{G}_{C^1,H^2}$ *to* $W^{1,\infty}([0,T), L^2(\mathbb{R}^3))$ *is the usual distributional derivative.*

(ii) *The same embedding turns* $C^1([0,T), H^\infty(\mathbb{R}^3))$ *into a subalgebra of* $\mathcal{G}_{C^1,H^2}$.

We introduce two more definitions relevant for equation (1).

Definition

Let $u \in \mathcal{G}_{C^1,H^2}$ with a representative $u_\varepsilon \in \mathcal{E}_{C^1,H^2}$. Since $u_\varepsilon \in C([0,T), H^2(\mathbb{R}^3))$, the function $u_\varepsilon(0,\cdot)$ is in $\mathcal{E}_{H^2}(\mathbb{R}^3)$. Also, if $u_\varepsilon \in \mathcal{N}_{C^1,H^2}$ then $u_\varepsilon(0,\cdot)$ is in $\mathcal{N}_{H^2}(\mathbb{R}^3)$. We define the restriction of u to $\{0\} \times \mathbb{R}^3$ as the class $[u_\varepsilon(0,\cdot)] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$.

Definition

We say that $a \in \mathcal{G}_{H^2}(\mathbb{R}^3)$ is of $(\ln)^j$–type, $j \in (0,1]$ if it has a representative $a_\varepsilon \in \mathcal{E}_{H^2}(\mathbb{R}^3)$ such that

$$\|a_\varepsilon\|_2 = \mathcal{O}(\ln^j \varepsilon^{-1}), \quad \varepsilon \to 0.$$

Note that a function $a \in H^\infty(\mathbb{R}^3)$ is itself a representative in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ and this is an example of a function that is of $(\ln)^j$–type for any $j \in (0,1]$.

Next we shortly describe the setting used in [4] for the fractional equation (2). We define

$$H^2_\alpha(\mathbb{R}^3) = \{\psi \in L^2(\mathbb{R}^3) |\ \psi = \phi_\lambda + \frac{\phi_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}} G_\lambda,\ \phi_\lambda \in H^2(\mathbb{R}^3)\},$$

$$(-\triangle_\alpha + \lambda)\psi = (-\triangle + \lambda)\phi_\lambda,$$

where $\lambda > 0$, $\alpha > 0$ are arbitrary fixed constants and

$$G_\lambda(x) := \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|}$$

is the Green's function for the Laplacian, that is, the distributional solution to $(-\triangle + \lambda)G_\lambda = \delta$ in $\mathcal{D}'(\mathbb{R}^3)$. Note that $G_\lambda \in L^2(\mathbb{R}^3)$.
The operator $\triangle_\alpha$ induces the Schrödinger propagator $t \mapsto e^{it\triangle_\alpha}$, analogous to the usual propagator. Norm on $H^2_\alpha(\mathbb{R}^3)$ is given by

$$\|\psi\|_{H^2_\alpha} = \|(I - \triangle_\alpha)\psi\|_2.$$

For arbitrary $\psi = \phi_\lambda + \frac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}} G_\lambda \in H^2_\alpha(\mathbb{R}^3)$ there holds

$$\|\psi\|_{H^2_\alpha} \approx \|\phi_\lambda\|_{H^2}.$$

A function u is a solution of (2) if $u \in C(I, H^2_\alpha(\mathbb{R}^3))$ for some interval $I \subset \mathbb{R}$ with $0 \in I$ and the Duhamel's formula

$$u(t) = e^{it\triangle_\alpha} a - i \int_0^t e^{i(t-s)\triangle_\alpha} (w * |u(s)|^2) u(s) ds \tag{5}$$

holds. This means that function u is a solution of (2) if $u \in C(I, H^2_\alpha(\mathbb{R}^3))$ is a fixed point for the map

$$\Phi(u)(t) = e^{it\triangle_\alpha} a - i \int_0^t e^{i(t-s)\triangle_\alpha} (w * |u(s)|^2) u(s) ds.$$

We define the notion of local well – posedness.

Definition

We say that the Cauchy problem (2) is locally well – posed in $H^2_\alpha(\mathbb{R}^3)$ if the following properties are satisfied

i) For any $a \in H^2_\alpha(\mathbb{R}^3)$ there exists $T_*, T^* \in (0, \infty]$ and a unique solution $u \in C((-T_*, T^*), H^2_\alpha(\mathbb{R}^3))$ of (5) and $(-T_*, T^*)$ is the maximal time interval where the solution is defined.

ii) The blowup alternative holds: if $T^* < \infty$ (respectively $T_* < \infty$) then $\lim_{t \to T^*} \|u(t)\|_{H^2_\alpha} = +\infty$ (resp. $\lim_{t \to T_*} \|u(t)\|_{H^2_\alpha} = +\infty$).

If $T^* = T_* = \infty$ then the solution is global. If there is local well – posedness and the solution is global we say that there is global well – posedness of (2). We analogously define well – posedness in some subspace V of $H^2_\alpha(\mathbb{R}^3)$. We are interested in connecting the Colombeau solution of (1) and the singular Sobolev solution of (2). With that purpose, we prove the following theorem.

Theorem

Let $w \in W^{2,p}(\mathbb{R}^3),\ p > 2$ and w is even. The Cauchy problem (2) is locally well-posed in the space

$$V = \{u \in H^2(\mathbb{R}^3), u \text{ is odd}\} \subset H^2(\mathbb{R}^3) \cap H^2_\alpha(\mathbb{R}^3)$$

and there is also global well – posedness.

Note that on the intersection of spaces $H^2(\mathbb{R}^3)$ and $H^2_\alpha(\mathbb{R}^3)$, the norms $\|\cdot\|_{H^2}$ and $\|\cdot\|_{H^2_\alpha}$ are equivalent and the characterization of this space is that $u \in H^2(\mathbb{R}^3)$ and $u(0) = 0$. The operator $-\triangle_\alpha$ acts as $-\triangle$ on the space of $H^2(\mathbb{R}^3)$ functions which vanish at zero.

Return now to the original equation:

$$\begin{aligned} iu_t + \triangle u - (w * |u|^2)u &= \delta u, \\ u(0,x) &= a(x). \end{aligned} \tag{6}$$

Definition

We say that $u \in \mathcal{G}_{C^1,H^2}$ is a solution of (6) if for an initial condition a, there exist representatives $u_\varepsilon \in \mathcal{E}_{C^1,H^2}$ and $a_\varepsilon \in \mathcal{E}_{H^2}(\mathbb{R}^3)$ such that

$$\begin{aligned} \sup_{t\in[0,T)} \|i(u_\varepsilon)_t + \triangle u_\varepsilon - (w * |u_\varepsilon|^2)u_\varepsilon - \phi_\varepsilon u_\varepsilon\|_2 &= O(\varepsilon^M), \quad \forall M \in \mathbb{N}, \\ u_\varepsilon(0,x) &= a_\varepsilon(x) + n_\varepsilon(x), \end{aligned} \tag{7}$$

where $n_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$ and ϕ_ε is a strict delta net.

If the above statement holds for some u_ε then it holds for all representatives of the class $u = [(u_\varepsilon)_\varepsilon]$.

Definition

We say that a solution of (6) is unique if for any two solutions $u, v \in \mathcal{G}_{C^1,H^2}$ there holds $\sup_{t\in[0,T)} \|u_\varepsilon - v_\varepsilon\|_2 = O(\varepsilon^M)$, for any $M \in \mathbb{N}$.

These definitions justify the use of spaces based on nets

$$u_\varepsilon \in C([0,T), H^2(\mathbb{R}^3)) \cap C^1([0,T), L^2(\mathbb{R}^3)),\ \varepsilon \in (0,1),$$

We consider regularized version of (6):

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon - (w * |u_\varepsilon|^2)u_\varepsilon = \phi_\varepsilon u_\varepsilon \\ u_\varepsilon(0,x) = a_\varepsilon(x), \end{aligned} \tag{8}$$

and $w \in W^{2,p}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ (due to the Sobolev embedding) and $a_\varepsilon \in \mathcal{E}_{H^2}(\mathbb{R}^3)$. We also assume that w is even.

Theorem

Let $a \in \mathcal{G}_{H^2}(\mathbb{R}^3)$ be of $\ln^{\frac{1}{3}}$–type. Then for any $T > 0$ there exists a unique solution $u \in \mathcal{G}_{C^1,H^2}$ of the equation (6).

Given the Cauchy problem (2) for $a \in V = \{u \in H^2(\mathbb{R}^3), u \text{ is odd}\}$, we know that there is a unique solution $u \in V$. Since $H^2(\mathbb{R}^3) \hookrightarrow \mathcal{G}_{H^2}(\mathbb{R}^3)$, for such an initial condition there is a unique solution of (6) in $\mathcal{G}_{C^1,H^2}$, also. This means there is a representative u_ε such that

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon - (w * |u_\varepsilon|^2)u_\varepsilon = \phi_\varepsilon u_\varepsilon \\ u_\varepsilon(0) = a_\varepsilon, \end{aligned} \tag{9}$$

for some regularization a_ε of a.
It does not mean that u_ε represents u (in order words, the two are not necessarily the same element of $\mathcal{G}_{C^1,H^2}$), but the following theorem holds.

Theorem

Let $a \in V$ and let u be the (fractional) Sobolev solution $u \in V$ of (2). *Let $[(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ be the Colombeau solution of* (6). *Then* $\sup_{[0,T)} \|u_\varepsilon(t) - u(t)\|_2 \to 0, \ \varepsilon \to 0$.

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