Abstract Friedrichs Operators

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Introduction

The concept of positive symmetric systems was introduced by Friedrichs, which are today customarily referred to as the Friedrichs systems. More precisely, for $d, r \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^d$ open and bounded with Lipschitz boundary, $\mathbf{A}_k \in W^{1,\infty}(\Omega; \mathrm{M}_r(\mathbb{C}))$, $k \in \{1, \ldots, d\}$, and $\mathbf{B} \in L^\infty(\Omega; \mathrm{M}_r(\mathbb{C}))$ satisfying (a.e. on Ω):

$$\mathbf{A}_k = \mathbf{A}_k^* ; \qquad \text{(F1)}$$

$$\exists \mu_0 > 0 \quad \mathbf{B} + \mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geq \mu_0 \mathbf{I} . \qquad \text{(F2)}$$

Define $\mathcal{L}, \widetilde{\mathcal{L}} : L^2(\Omega)^r \to \mathcal{D}'(\Omega)^r$ by

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{B}\mathsf{u} ,$$

$$\widetilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \Big(\mathbf{B}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k\Big)\mathsf{u} ,$$

is called Classical Friedrichs System.

Aim: to impose boundary conditions such that for any $\mathsf{f} \in L^2(\Omega)^r$ we have a unique solution of $\mathcal{L}\mathsf{u} = \mathsf{f}$.

Gain: many important (semi)linear equations of mathematical physics can be written in the form of classical Friedrichs operators.

Cassical theory in short: Unified treatment of linear hyperbolic systems like Maxwell's, Dirac's, or higher order equations (e.g. the wave equation).

– treating the equations of mixed type, such as the Tricomi equation:

$$y\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ;$$

– unified treatment of equations and systems of different type;

– more recently: better numerical properties.

Shortcommings:

– no satisfactory well-posedness result,

– no intrinsic (unique) way to pose boundary conditions.

⇝ development of the abstract theory

$(\mathcal{H}, \langle \cdot \mid \cdot \rangle)$ complex Hilbert space ($\mathcal{H}' \equiv \mathcal{H}$), $\|\cdot\| := \sqrt{\langle \cdot \mid \cdot \rangle}$, $\mathcal{D} \subseteq \mathcal{H}$ dense subspace. Let Let $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$. The pair $(T, \widetilde{T})$ is called a joint pair of abstract Friedrichs operators if the following holds:

$$(\forall \varphi, \psi \in \mathcal{D}) \qquad \langle T\varphi \mid \psi \rangle = \langle \varphi \mid \widetilde{T}\psi \rangle ; \qquad \text{(T1)}$$

$$(\exists\, c > 0)(\forall \varphi \in \mathcal{D}) \qquad \|(T + \widetilde{T})\varphi\| \leqslant c\|\varphi\| ; \qquad \text{(T2)}$$

$$(\exists\, \mu_0 > 0)(\forall \varphi \in \mathcal{D}) \qquad \langle (T + \widetilde{T})\varphi \mid \varphi \rangle \geqslant \mu_0\|\varphi\|^2 . \qquad \text{(T3)}$$

Note: Classical is abstract.

Characterisation of joint pair of abstract Friedrichs operators

Lemma

$$(T1) - (T3) \iff \begin{cases} T \subseteq \widetilde{T}^* \quad \& \quad \widetilde{T} \subseteq T^*; \\ \overline{T + \widetilde{T}} \text{ bounded self-adjoint in } \mathcal{H} \\ \text{with strictly positive bottom;} \\ \operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} \quad \& \quad \operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* . \end{cases}$$

By (T1) , T and $\widetilde{T}$ are closable. By (T2) , $T+\widetilde{T}$ is a bounded operator, so the graph norms $\|\cdot\|_T$ and $\|\cdot\|_{\widetilde{T}}$ are equivalent.

$$\operatorname{dom} \overline{T} = \operatorname{dom} \overline{\widetilde{T}} =: \mathcal{W}_0 , \qquad \operatorname{dom} T^* = \operatorname{dom} \widetilde{T}^* =: \mathcal{W} , \qquad (1)$$

and $\overline{(T + \widetilde{T})}|_{\mathcal{W}} = \widetilde{T}^* + T^*$. So, $(\overline{T}, \overline{\widetilde{T}})$ is also a pair of abstract Friedrichs operators.

Notation :

$$T_0 := \overline{T} , \quad \widetilde{T}_0 := \overline{\widetilde{T}} , \quad T_1 := \widetilde{T}^* , \quad \widetilde{T}_1 := T^* .$$

Therefore, we have

$$T_0 \subseteq T_1 \quad \text{and} \quad \widetilde{T}_0 \subseteq \widetilde{T}_1 . \qquad (2)$$

$(\mathcal{W}, \|\cdot\|_T)$ is the *graph space*. $\mathcal{W}_0$ is a closed subspace of the graph space $\mathcal{W}$.

For, $\mathcal{D} = C_c^\infty(\Omega)$, $\mathcal{H} = L^2(\Omega)$ and a certain choice of operators it could be that $\mathcal{W}$ and $\mathcal{W}_0$ are Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively.

Boundary map (form): $D : \mathcal{W} \to \mathcal{W}'$,

$$[\, u \mid v \,] := {}_{\mathcal{W}'}\langle Du, v \rangle_{\mathcal{W}} := \langle T_1 u \mid v \rangle - \langle u \mid \widetilde{T}_1 v \rangle .$$

Let a pair of operators $(T, \widetilde{T})$ on $\mathcal{H}$ satisfies (T1)–(T2). Then D is continuous and satisfies

i) $(\forall u, v \in \mathcal{W}) \qquad ([\, u \mid v \,] = \overline{[\, v \mid u \,]})$,

ii) $\ker D = \mathcal{W}_0$.

Remark: $(\mathcal{W}, [\cdot \mid \cdot])$ is indefinite inner product space.

Well-posedness Result

For $\mathcal{V}, \widetilde{\mathcal{V}} \subseteq \mathcal{W}$ we introduce two conditions:

$$\text{(V1)} \qquad (\forall\, u \in \mathcal{V}) \quad [\, u \mid u \,] \geqslant 0 , \qquad (\forall\, v \in \widetilde{\mathcal{V}}) \quad [\, v \mid v \,] \leqslant 0$$

$$\text{(V2)} . \qquad \mathcal{V}^{[\perp]} = \widetilde{\mathcal{V}} , \; \widetilde{\mathcal{V}}^{[\perp]} = \mathcal{V}$$

Theorem[Ern, Guermond, Caplain, 2007]
(T1)–(T3) + (V1)–(V2) $\implies$ $T_1|_{\mathcal{V}}, \widetilde{T}_1|_{\widetilde{\mathcal{V}}}$ bijective realisations .

Existence, Multiplicity and Classification

We seek for bijective closed operators $S \equiv \widetilde{T}^*|_{\mathcal{V}}$ such that

$$\overline{T} \subseteq S \subseteq \widetilde{T}^* ,$$

and thus also S^* is bijective and $\overline{\widetilde{T}} \subseteq S^* \subseteq T^*$. We call (S, S^*) an adjoint pair of bijective realisations relative to $(T, \widetilde{T})$.

Theorem[Antonić, Erceg, Michelangeli, 2017]
Let $(T, \widetilde{T})$ satisfies (T1)–(T3).

(i) **Existence**: There exists an adjoint pair of bijective realisations with signed boundary map relative to $(T, \widetilde{T})$.

(ii) **Multiplicity**:

$\ker \widetilde{T}^* \neq \{0\}$ & $\ker T^* \neq \{0\}$ $\implies$ uncountably many adjoint pairs of bijective realisations with signed boundary map

$\ker \widetilde{T}^* = \{0\}$ or $\ker T^* = \{0\}$ $\implies$ only one adjoint pair of bijective realisations with signed boundary map

Classification: For $(T, \widetilde{T})$ satisfying (T1)–(T3) we have

$$\overline{T} \subseteq \widetilde{T}^* \qquad \text{and} \qquad \overline{\widetilde{T}} \subseteq T^* ,$$

while by the previous theorem there exists closed T_{r} such that

- $\overline{T} \subseteq T_{\mathrm{r}} \subseteq \widetilde{T}^*$ ($\iff$ $\overline{\widetilde{T}} \subseteq T_{\mathrm{r}}^* \subseteq T^*$),
- $T_{\mathrm{r}} : \operatorname{dom} T_{\mathrm{r}} \to \mathcal{H}$ bijection,
- $(T_{\mathrm{r}})^{-1} : \mathcal{H} \to \operatorname{dom} T_{\mathrm{r}}$ bounded.

Thus, we can apply Grubb's universal classification theory (classification of dual (adjoint) pairs).

Result: complete classification of all adjoint pairs of bijective realisations with signed boundary map.

To do: apply this result to general classical Friedrichs operators from the beginning.

Decomposition of the graph space

Theorem[Erceg, Soni, 2022]
$(T_0, \widetilde{T}_0)$ is a joint pair of closed abstract Friedrichs operators then

$$\mathcal{W} = \mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1 .$$

Corollary: $\Big(T_1|_{\mathcal{W}_0 \dot{+} \ker \widetilde{T}_1}, \widetilde{T}_1|_{\mathcal{W}_0 \dot{+} \ker T_1}\Big)$ is a pair of mutually adjoint pair of bijective realisations relative to $(T, \tilde{T})$.

A sketch for the proof of the theorem is:

- $\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1$ is direct and closed in $\mathcal{W}$.
- For any bijective realisation T_{r},
  $$\mathcal{W} = \mathcal{W}_0 \dot{+} T_{\mathrm{r}}^{-1}(\ker \widetilde{T}_1) \dot{+} \ker T_1 = \mathcal{W}_0 \dot{+} (T_{\mathrm{r}}^*)^{-1}(\ker T_1) \dot{+} \ker \widetilde{T}_1 .$$
- $\mathcal{W} = \Big(\mathcal{W}_0 \dot{+} \ker T_1 \dot{+} \ker \widetilde{T}_1\Big)^{[\perp][\perp]}$.

Using the above theorem we now find all admissible boundary conditions for 1-d scalar case with variable coefficients.

One-dimensional($d = 1$) Scalar($r = 1$) Case

$\Omega = (a, b)$, $a < b$, $\mathcal{D} = C_c^\infty(a, b)$ and $\mathcal{H} = L^2(a, b)$. $T, \widetilde{T} : \mathcal{D} \to \mathcal{H}$:

$$T\varphi := (\alpha\varphi)' + \beta\varphi \qquad \text{and} \qquad \widetilde{T}\varphi := -(\alpha\varphi)' + (\overline{\beta} + \alpha')\varphi .$$

Here $\alpha \in W^{1,\infty}((a, b); \mathbb{R})$, $\beta \in L^\infty((a, b); \mathbb{C})$ and for some $\mu_0 > 0$, $2\Re\beta + \alpha' \geq 2\mu_0 > 0$.

The graph space :

$$\mathcal{W} = \{u \in \mathcal{H} : (\alpha u)' \in \mathcal{H}\} , \qquad \|u\|_{\mathcal{W}} := \|u\| + \|(\alpha u)'\| .$$

Equivalently ,

$$u \in \mathcal{W} \iff \alpha u \in H^1(a, b) .$$

So, by Sobolev embedding $\alpha u \in C(a, b)$. Implies the evaluation $(\alpha u)(x)$ is well defined. However, u is not necessarily continuous so $\alpha(x)u(x)$ is not meaningful.

Lemma Let $I := [a, b] \setminus \alpha^{-1}(\{0\})$. Then $\mathcal{W} \subseteq H^1_{\mathrm{loc}}(I)$, i.e. for any $u \in \mathcal{W}$ and $[c, d] \subseteq I$, $c < d$, we have $u|_{[c,d]} \in H^1(c, d)$.

The boundary operator can be written explicitly as

$${}_{\mathcal{W}'}\langle\, Du, v \,\rangle_{\mathcal{W}} = \big(\alpha u\overline{v}\big)(b) - \big(\alpha u\overline{v}\big)(a) , \quad u, v \in \mathcal{W} ,$$

where we define

$$\big(\alpha u\overline{v}\big)(x) := \begin{cases} 0 & , \; \alpha(x) = 0 \\ \alpha(x)u(x)\overline{v(x)} & , \; \alpha(x) \neq 0 \end{cases} , \qquad x \in [a, b] .$$

The domain of the closures T_0 and $\widetilde{T}_0$ satisfies $\mathcal{W}_0 = \mathrm{cl}_{\mathcal{W}} C_c^\infty(\mathbb{R})$, is characterised as

Lemma

$$\mathcal{W}_0 = \Big\{u \in \mathcal{W} : (\alpha u)(a) = (\alpha u)(b) = 0\Big\} .$$

Lemma The codimension of the quotient space $\mathcal{W}/\mathcal{W}_0$ is

$$= \begin{cases} 2 \;, \; \alpha(a)\alpha(b) \neq 0 \;, \\ 1 \;, \; \big(\alpha(a) = 0 \wedge \alpha(b) \neq 0\big) \vee \big(\alpha(a) \neq 0 \wedge \alpha(b) = 0\big) \\ 0 \;, \; \alpha(a) = \alpha(b) = 0 \;. \end{cases}$$

By the decomposition we have

$$\dim(\ker T_1) + \dim(\ker \widetilde{T}_1) = \dim \mathcal{W}/\mathcal{W}_0 .$$

Thus, when $\alpha(a)\alpha(b) = 0$ there is only one bijective realisation of T_0.
In case $\alpha(a)\alpha(b) \neq 0$ there are infinitely many bijective realisations if and only if $\dim(\ker T_1) = \dim(\ker \widetilde{T}_1)$.
The only interesting case is, when $\alpha(a) > 0, \alpha(b) > 0$. In this case we have,
$u \in \mathcal{W}$ belongs to $\operatorname{dom} T_{c,d}$ if and only if

$$[1] \quad \left(\frac{\alpha(b)\overline{\tilde{\varphi}(b)}}{\|\tilde{\varphi}\|^2} - \frac{(c + id)}{\varphi(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)}\right) u(b) = \left(\frac{\alpha(a)\overline{\tilde{\varphi}(a)}}{\|\tilde{\varphi}\|^2} - \frac{(c + id)\sqrt{\frac{\alpha(a)}{\alpha(b)}}}{\varphi(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\varphi(a)}\right) u(a) .$$

Similarly, $u \in \mathcal{W}$ is in $\operatorname{dom} T^*_{c,d}$ if and only if

$$[2] \quad \left(\alpha(b)\overline{\varphi(b)} - \frac{\|\tilde{\varphi}\|^2(c - id)}{\tilde{\varphi}(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\tilde{\varphi}(a)}\right) u(b) = \left(\alpha(a)\overline{\varphi(a)} - \frac{\|\tilde{\varphi}\|^2(c - id)\sqrt{\frac{\alpha(a)}{\alpha(b)}}}{\tilde{\varphi}(b) - \sqrt{\frac{\alpha(a)}{\alpha(b)}}\tilde{\varphi}(a)}\right) u(a) .$$

So, the set of all pairs of mutually adjoint bijective realisations relative to $(T, \widetilde{T})$ is given by

$$[3] \quad \Big\{(T_{c,d}, T^*_{c,d}) : c, d \in \mathbb{R}^2 \setminus \{(0,0)\}\Big\} \bigcup \big\{(T_{\mathrm{r}}, T_{\mathrm{r}}^*)\big\} .$$

Summary :

α at end-points	No. of bij. realis.	$(\mathcal{V}, \widetilde{\mathcal{V}})$	
$\alpha(a)\alpha(b) \leq 0$	1	$\alpha(a) \geq 0 \wedge \alpha(b) \leq 0$	$(\mathcal{W}_0, \mathcal{W})$
		$\alpha(a) \leq 0 \wedge \alpha(b) \geq 0$	$(\mathcal{W}, \mathcal{W}_0)$
$\alpha(a)\alpha(b) > 0$	∞	[3] (see [1] and [2])	

Acknowledgements

This work is supported by Croatian Science Foundation under the project IP-2018-01-2449 MiTPDE.

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